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Diagram exhibiting the division of a pulse into two undulations wi

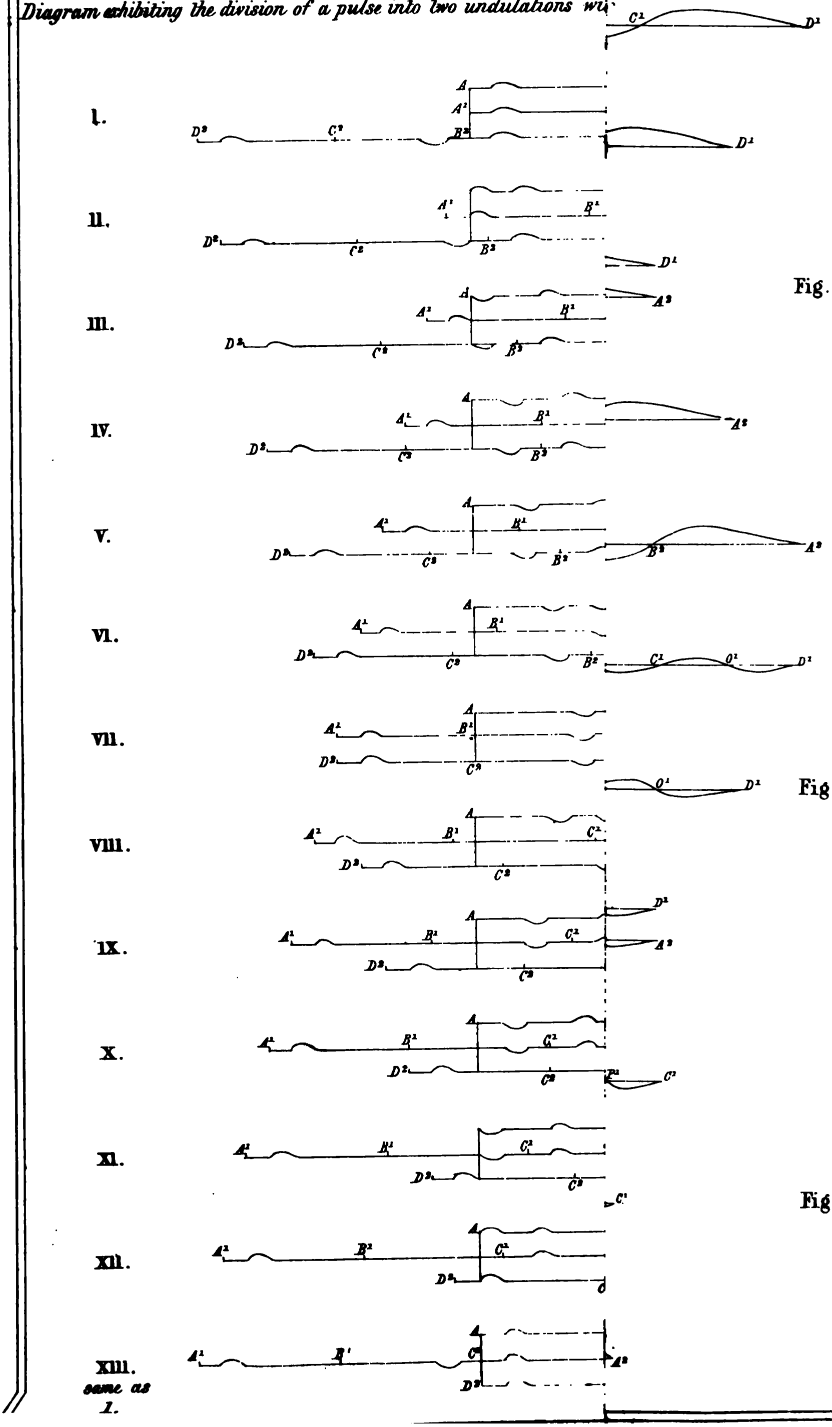

Fig. 1.

Fig. 2.

Fig. 3.

THE

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VOL. I.

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INDEX TO VOL. I.

PLANE GEOMETRY.

ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

* The numbers in brackets indicate pages of the original edition.

ALGEBRA.

DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

ASTRONOMY.

LIGHT AND SOUND.

MECHANICS.

HYDROSTATICS.

GENERAL INDEX TO THE FIRST SERIES.

GEOMETRY OF TWO DIMENSIONS.

* In references to vol. i. the pages of the first edition are indicated.

GEOMETRY OF THREE DIMENSIONS.

ALGEBRA AND TRIGONOMETRY.

DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

MECHANICS.

ASTRONOMY.

LIGHT AND SOUND.

HEAT, ELECTRICITY, AND MAGNETISM.

MISCELLANEOUS.

PREFACE.

In bringing before the Public the First Number of the *Cambridge Mathematical Journal*, it will naturally be expected that we should say a few words, in the way of preface, on the objects we have in view, and the means we intend to adopt for carrying them into effect.

It has been a subject of regret with many persons, that no proper channel existed, either in this University or elsewhere in this country, for the publication of papers on Mathematical subjects, which did not appear to be of sufficient importance to be inserted in the Transactions of any of the Scientific Societies; the two Philosophical Journals which do exist having their pages generally devoted to physical subjects. In this place in particular, where the Mathematics are so generally cultivated, it might be expected that there would be an opening for a work exclusively devoted to that science which does not command much interest in the world at large. We think that there can be no doubt that there are many persons here who are both able and willing to communicate much valuable matter to a Mathematical periodical, while the very existence of such a work is likely to draw out others, and make them direct their attention in some degree to original research. Our primary object, then, is to supply a means of publication for original papers.

But we conceive that our Journal may likewise be rendered useful in another way—by publishing abstracts of

important and interesting papers that have appeared in the Memoirs of foreign Academies, and in works not easily accessible to the generality of students. We hope in this way to keep our readers, as it were, on a level with the progressive state of Mathematical science, and so lead them to feel a greater interest in the study of it. For this purpose we shall spare no pains in selecting the most useful and important papers from which to take abstracts for the benefit of our readers, while we shall put them in such a form as to make them available in the studies of this place. At the same time we shall endeavour always to have such a variety of subjects treated of, that all classes of students may find in our journal something which may be useful to them.

Our intention is to publish only one Number in each term, which for many reasons we think preferable to a more frequent appearance. Those gentlemen who intend to favour us with their communications, are requested to send them, addressed to the Editors of the Journal, to our publishers.

Trinity College, Oct. 1837.

THE CAMBRIDGE

MATHEMATICAL JOURNAL.

NOTES ON THE UNDULATORY THEORY OF LIGHT. NO. I.

THE following demonstrations of some Theorems in Fresnel's Memoir on Double Refraction, are not offered as part of a complete treatise on this part of the Undulatory Theory, but merely as shewing that some of the remarkable propositions deduced by that author may be conveniently proved by a shorter and more direct analysis than that which he has employed. Fresnel generally makes use of a mixed Geometry, which was perhaps the best method for establishing his theorems at first; but as his proofs are often tedious, it seems desirable to obtain demonstrations more suited to the general style of mathematics in researches of this kind.

1. We shall begin with a demonstration of the existence of a system of three axes of elasticity—a proposition on which the whole theory of double refraction depends, and which Fresnel has proved by a method which has the advantage of geometrical distinctness, but which is long and on that account rather difficult to follow out. The proposition is thus stated:—*In any system of particles acting on each other with forces which are functions of their mutual distances, there are three directions at right angles to each other, along which if a particle be displaced, the forces of restitution will act in the same direction.*

Let x, y, z, be the coordinates of the attracted point,

x_1, y_1, z_1, x_2, y_2, z_2, ... be the coordinates of the attracting points,

r_1, r_2, r_3, ... the distances between the attracted and attracting points,

$\phi_1(r_1)$, $\phi_2(r_2)$, $\phi_3(r_3)$, ... the attractions,

X, Y, Z, the total resolved forces along the axes;

then we shall have

$$X = \frac{x_1 - x}{r_1}\,\phi_1(r_1) + \frac{x_2 - x}{r_2}\,\phi_2(r_2) + \ldots.$$

And similarly for Y and Z. Now let

$$R = -\Sigma \int \phi(r)\,dr.$$

Then
$$\left.\begin{aligned} X &= \frac{dR}{dx} = 0,\\ Y &= \frac{dR}{dy} = 0,\\ Z &= \frac{dR}{dz} = 0, \end{aligned}\right\}\ \text{when the particle is in equilibrio.}$$

Let the particle receive a small displacement, the projections of which on the coordinate axes are δx, δy, δz. Then, supposing the displacement to be very small, we have

$$X = \frac{d^2R}{dx^2}\,\delta x + \frac{d^2R}{dx\,dy}\,\delta y + \frac{d^2R}{dx\,dz}\,\delta z,$$

$$Y = \frac{d^2R}{dy\,dx}\,\delta x + \frac{d^2R}{dy^2}\,\delta y + \frac{d^2R}{dy\,dz}\,\delta z,$$

$$Z = \frac{d^2R}{dz\,dx}\,\delta x + \frac{d^2R}{dz\,dy}\,\delta y + \frac{d^2R}{dz^2}\,\delta z.$$

Now the force of restitution will be in the direction of the displacement, if X, Y, Z, be proportional to δx, δy, δz. Let then

$$s = \frac{X}{\delta x} = \frac{Y}{\delta y} = \frac{Z}{\delta z}.$$

Then putting

$$\frac{d^2R}{dx^2} = A, \qquad \frac{d^2R}{dy^2} = B, \qquad \frac{d^2R}{dz^2} = C,$$

$$\frac{d^2R}{dz\,dy} = \frac{d^2R}{dy\,dz} = A', \quad \frac{d^2R}{dz\,dx} = \frac{d^2R}{dx\,dz} = B', \quad \frac{d^2R}{dx\,dy} = \frac{d^2R}{dy\,dx} = C';$$

and substituting in the former equations, they become

$$\left.\begin{aligned} (A - s)\,\delta x + C'\delta y + B'\delta z &= 0,\\ C'\delta x + (B - s)\,\delta y + A'\delta z &= 0,\\ B'\delta x + A'\delta y + (C - s)\,\delta z &= 0, \end{aligned}\right\}\ldots\ldots (a).$$

Eliminating δx, δy, δz, by cross multiplication,* we obtain, as an equation of condition,

$$(A - s)(B - s)(C - s) - A'^2(A - s) - B'^2(B - s) - C'^2(C - s) + 2A'B'C' = 0.$$

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* For an explanation of this method, see Article, p. [46].

This is obviously the same equation as that found in investigating the existence of three principal diametral planes in surfaces of the second order, as well as of the three prin- [5]
cipal axes of rotation. As it is a cubic equation, it has at least one real root. Let us suppose that the axis of z is that which corresponds to this root, so that a displacement along it produces a force of restitution acting in the same direction. In this case A' and B' will vanish, for A' is the force along the axis of y arising from a displacement along z, and B' is the corresponding quantity for x. The equations are thus reduced to

$$(A - s) + C'\frac{\delta y}{\delta x} = 0,$$

$$(B - s)\frac{\delta y}{\delta x} + C' = 0.$$

Eliminating s, we get

$$\left(\frac{\delta y}{\delta x}\right)^2 + \frac{A - B}{C'}\frac{\delta y}{\delta x} - 1 = 0.$$

The last term in this quadratic being -1, the two lines whose directions are determined by the two values of $\frac{\delta y}{\delta x}$ are at right angles to each other, and as $\delta z = 0$, they are in the plane of xy; consequently there are three directions at right angles to each other, along which, if a particle be displaced, the force of restitution acts in the same direction.*

* [Cauchy has shewn in the following manner that the three directions, determined by the preceding equations, are at right angles to one another.

Let s_1 and s_2 be any two roots of the cubic equation, and let δx_1, δy_1, δz_1, and δx_2, δy_2, δz_2 be the components of displacements in the directions corresponding to these roots. Equations (a) must be satisfied by either system of values of s, δx, δy, δz, and therefore we have

$$s_1\,\delta x_1 = A\delta x_1 + C''\delta y_1 + B'\delta z_1,$$
$$s_1\,\delta y_1 = C''\delta x_1 + B\delta y_1 + A'\delta z_1,$$
$$s_1\,\delta z_1 = B'\delta x_1 + A'\delta y_1 + C\delta z_1.$$

Hence $$s_1\,(\delta x_1\delta x_2 + \delta y_1\delta y_2 + \delta z_1\delta z_2) = Q,$$

where $Q = A\delta x_1\delta x_2 + B\delta y_1\delta y_2 + C\delta z_1\delta z_2$

$$+ A'(\delta y_1\delta z_2 + \delta z_1\delta y_2) + B'(\delta z_1\delta x_2 + \delta x_1\delta z_2) + C''(\delta x_1\delta y_2 + \delta y_1\delta x_2).$$

Similarly, by substituting in equations (a) the second system of values for s, δx, δy, δz, we should have found, on account of the symmetry of Q with respect to δx_1, δy_1, δz_1, and δx_2, δy_2, δz_2,

$$s_2\,(\delta x_2\delta x_1 + \delta y_2\delta y_1 + \delta z_2\delta z_1) = Q.$$

Hence $$(s_1 - s_2)\,(\delta x_1\delta x_2 + \delta y_1\delta y_2 + \delta z_1\delta z_2) = 0,$$

2. The next proposition we shall prove, is that for determining the velocities of the waves of light in a crystal. But it will be necessary first to recal Fresnel's construction for finding them.

Having proved, as we have just done, that in every crystal there are three axes of elasticity passing through every point, it is a natural supposition, confirmed by experiment, that these axes are always parallel to fixed straight lines. Take therefore these axes as the coordinate axes, and let the forces excited by displacements equal to unity in these directions be a^2, b^2, c^2, respectively. Then if a particle receive a displacement $= 1$ in a direction making angles X, Y, Z, with these axes, the resolved parts of the displacement will be

$$\cos X, \; \cos Y, \; \cos Z,$$

and the resolved parts of the force will be

$$a^2 \cos X, \; b^2 \cos Y, \; c^2 \cos Z;$$

so that if f be the whole force,

$$f = \sqrt{(a^4 \cos^2 X + b^4 \cos^2 Y + c^4 \cos^2 Z)},$$

and the cosines of the angles which its direction makes with the axes are

$$\frac{a^2 \cos X}{f}, \; \frac{b^2 \cos Y}{f}, \; \frac{c^2 \cos Z}{f},$$

and the cosine of the angle between the direction of displacement and the direction of the force of restitution will be

$$\frac{a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z}{f}.$$

[6] And if the force be resolved along, and perpendicular to, the direction of the displacement, the former part will be

$$a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z.$$

If now we construct a surface whose equation is

$$r^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z,$$

and a particle be displaced along any radius, the square of

---

and therefore, if s_1 and s_2 be different,

$$\delta x_1 \delta x_2 + \delta y_1 \delta y_2 + \delta z_1 \delta z_2 = 0,$$

which shews that any two directions of displacement corresponding to two different roots of the cubic, are at right angles. Since the cubic has necessarily three real roots (for a demonstration of this theorem see p. [275]), it follows that there are three, and in general only three, directions possessing the required property, and that they are at right angles to one another. For another method of treating equations (a), see vol. IV. p. 227].

that radius will represent the resolved part, in that direction, of the force of restitution. This surface is called the surface of elasticity, and its equation between rectangular coordinates evidently is

$$a^2x^2 + b^2y^2 + c^2z^2 = r^4 = (x^2 + y^2 + z^2)^2.$$

When the part of the force which is perpendicular to the direction of displacement is also perpendicular to the front of the wave, Fresnel has shewn that it will produce no effect, and therefore may be neglected. But this will not generally be the case, and therefore this force will be equivalent to two others—one perpendicular to the front of the wave, which may be neglected, and another in the plane of the front of the wave, which will have to be compounded with that in the direction of the displacement; so that, in general, the total effective force will not be in the direction of the displacement. It will be shewn, however, that in every plane there are two directions at right angles to each other, in which, if a particle be displaced, the part of the force perpendicular to the line of displacement will also be perpendicular to the plane. If a displacement take place in any other direction in that plane, we may resolve it in those two directions, so that the forces excited will be in the same directions and proportional to the squares of the corresponding radii of the surface of elasticity, and vibrations parallel to these directions will traverse the medium with velocities proportional to the radii.

To prove the existence of these two directions, let

$$lx + my + nz = 0$$

be the equation to the plane of the front of the wave,

$$l'x + m'y + n'z = 0$$

the equation to a plane passing through the directions of the displacement and of the excited force, so that

$$l' \cos X + m' \cos Y + n' \cos Z = 0 \dots\dots (1),$$

and $$l'a^2 \cos X + m'b^2 \cos Y + n'c^2 \cos Z = 0 \dots\dots (2).$$

In order that the part of the force which is not in the direction of the displacement may be perpendicular to the front of the wave, these two planes must be perpendicular to each other; therefore

$$ll' + mm' + nn' = 0 \dots\dots\dots\dots (3).$$

Eliminating l', m', n', between these three equations, by cross multiplication, we get

$$(b^2 - c^2)\, l \cos Y \cos Z + (c^2 - a^2)\, m \cos Z \cos X$$
$$+ (a^2 - b^2)\, n \cos X \cos Y = 0 \dots\dots (4);$$

[7] which, together with the equations

$$l \cos X + m \cos Y + n \cos Z = 0 \ldots\ldots (5),$$

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1 \ldots\ldots (6),$$

determine the angles X, Y, Z, and therefore the direction of the displacement.

These directions are the same as those of the greatest and least radii of a section of the surface of elasticity made by the same plane. For to determine these we have the equation

$$r^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z,$$

with the condition $dr = 0$, and the equations (5) and (6).

Differentiating, we get

$$a^2 \cos X\, d \cos X + b^2 \cos Y\, d \cos Y + c^2 \cos Z\, d \cos Z = 0,$$

$$l\, d \cos X + m\, d \cos Y + n\, d \cos Z = 0,$$

$$\cos X\, d \cos X + \cos Y\, d \cos Y + \cos Z\, d \cos Z = 0.$$

Eliminating, as before, $d \cos X$, $d \cos Y$, $d \cos Z$, between these three equations, we get

$$(b^2 - c^2)\, l \cos Y \cos Z + (c^2 - a^2)\, m \cos Z \cos X + (a^2 - b^2)\, n \cos X \cos Y = 0;$$

the same as (4), so that X, Y, Z, being determined by the same equations in the two cases, the resulting values will be the same in both. Hence, in order to find the velocities of the waves of light in passing through a crystal, we have merely to determine the greatest and least radii of a section of the surface of elasticity. This may be effected more readily than is done by Fresnel, in the following manner.

Let $$r^4 = a^2x^2 + b^2y^2 + c^2z^2 \ldots\ldots\ldots\ldots (1)$$

be the equation to the surface of elasticity, where

$$r^2 = x^2 + y^2 + z^2 \ldots\ldots\ldots\ldots\ldots (2),$$

and let the surface be cut by a plane

$$0 = lx + my + nz \ldots\ldots\ldots\ldots\ldots (3).$$

When r is the greatest or least radius in the section made by this plane, we have the condition $dr = 0$. Differentiating the three equations with this condition, we get

$$0 = a^2x\, dx + b^2y\, dy + c^2z\, dz \ldots\ldots\ldots (4),$$

$$0 = x dx + y\, dy + z\, dz \ldots\ldots\ldots\ldots (5),$$

$$0 = l\, dx + m\, dy + n\, dz \ldots\ldots\ldots\ldots (6).$$

Then $A\,(6) + B\,(5) + (4)$ gives, on equating to 0 the coefficients of each differential,

$$Al + Bx + a^2x = 0,$$
$$Am + By + b^2y = 0,$$
$$An + Bz + c^2z = 0.$$

Multiply by x, y, z, and add, considering the conditions (1), (2), (3). Then we have

$$Br^2 + r^4 = 0, \quad \text{or } B = -r^2;$$

therefore, substituting [8]

$$Al = (r^2 - a^2)\,x, \quad Am = (r^2 - b^2)\,y, \quad An = (r^2 - c^2)\,z,$$

or $$x = \frac{Al}{r^2 - a^2}, \quad y = \frac{Am}{r^2 - b^2}, \quad z = \frac{An}{r^2 - c^2}.$$

Multiply by l, m, n, and add; then, by the condition (3), we have

$$\frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0,$$

a quadratic equation in r^2 from which two values of r^2 may be found, and thus the velocities determined.

It is easy from this to determine the equation to the wave surface, for it is the locus of the ultimate intersections of planes the perpendiculars on which from the origin are determined by the above equation. Calling the perpendicular v, it will therefore be determined by the following equations:

$$lx + my + nz = v \ldots\ldots\ldots\ldots\ldots(1),$$

and $$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \ldots\ldots\ldots\ldots\ldots(2),$$

where $$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots\ldots(3).$$

For the process employed the reader is referred to the *Cambridge Transactions*, vol. VI. part I. But we may add here a method of finally eliminating l, m, n, and v, which is somewhat shorter than that employed there.

Having found that

$$x\,(a^2 - v^2) = lv\,(a^2 - r^2), \qquad y\,(b^2 - v^2) = mv\,(b^2 - r^2),$$
$$z\,(c^2 - v^2) = nv\,(c^2 - r^2),$$

substitute the values of l, m, n, given by these equations in (1), which then becomes

$$\frac{x^2\,(a^2 - v^2)}{a^2 - r^2} + \frac{y^2\,(b^2 - v^2)}{b^2 - r^2} + \frac{z^2\,(c^2 - v^2)}{c^2 - r^2} = v^2 \ldots.(4);$$

also, we have $$x^2 + y^2 + z^2 = r^2 \ldots\ldots\ldots\ldots\ldots(5).$$

Subtracting (4) from (5), we get

$$\left(\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}\right)(v^2-r^2)=-(v^2-r^2),$$

or
$$\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}+1=0\ldots\ldots\ldots(6),$$

which is one form of the equation to the wave surface. The form used by Fresnel may be easily deduced by combining equations (4) and (6). For if we split each term of (4), it becomes

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}-v^2\left\{\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}\right\}=v^2,$$

[9] which by the condition (6) reduces itself to

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0;$$

in which, if we substitute $x^2+y^2+z^2$ for r^2, and multiply up, we shall obtain Fresnel's form of the equation.

A. S.

ON THE EQUATION TO THE TANGENT OF THE ELLIPSE.

In the usual method of finding the Equation to the tangent of an Ellipse, the point of contact is given so that the required equation involves its coordinates; and if we wish to deduce any general properties of the tangent independent of the particular point of contact, we have to eliminate two quantities, which necessarily renders the operation of elimination troublesome. This may be avoided by finding the condition that a line should be a tangent to an ellipse without specifying the point of contact. For by this means only one indeterminate quantity is introduced, the elimination of which is generally easily effected. The quantity chosen is the tangent of the angle which the tangent to the curve makes with the axis of x.

Let
$$y=\alpha x+\beta\ldots\ldots\ldots\ldots\ldots(1)$$

be the equation to a line cutting the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ldots\ldots\ldots\ldots\ldots(2).$$

If we substitute the value of y from (1) in (2) we obtain a quadratic equation in x, the roots of which are the values of x at the two points where the line in general cuts the ellipse.

If the line become a tangent, these two values of x must be equal; and on making the condition that the quadratic in x shall be a complete square, we obtain, as an equation of condition,

$$\beta = \sqrt{(a^2\, \alpha^2 + b^2)}.$$

Hence the equation to the tangent to the ellipse may be put under the form

$$y = \alpha x + \sqrt{(a^2\, \alpha^2 + b^2)} \ldots\ldots\ldots\ldots (3),$$

where α is the tangent of the angle which the line makes with the axis of x.

This form of the equation is not new, for it is given by Mr. Waud, in p. 73 of his Algebraical Geometry. But that author does not seem to have observed its use in de- [10]
ducing several of the principal properties of the Ellipse. Some of these applications we shall now give.

1. To find the locus of the intersection of two tangents to an ellipse, which are at right angles to each other.

By (3) the equation to the one tangent is

$$y = \alpha x + \sqrt{(a^2\, \alpha^2 + b^2)};$$

the equation to the other, which is perpendicular to it, is

$$y = -\frac{1}{\alpha}\, x + \sqrt{\left(\frac{a^2}{\alpha^2} + b^2\right)},$$

or

$$\alpha y = -x + \sqrt{(a^2 + b^2\, \alpha^2)}.$$

Transposing, squaring, and adding the two equations, we have

$$(1 + \alpha^2)\,(x^2 + y^2) = (a^2 + b^2)\,(1 + \alpha^2),$$

whence

$$x^2 + y^2 = a^2 + b^2.$$

2. To find the product of the perpendiculars from the foci on the tangent.

The coordinates of the foci are

$$x = ae,\ y = 0, \qquad x = -\,ae,\ y = 0.$$

Therefore the length of the perpendiculars p_1, p_2, on the tangent

$$y = \alpha x + \sqrt{(a^2\, \alpha^2 + b^2)},$$

are

$$p_1 = -\,\frac{\alpha ae + \sqrt{(a^2\, \alpha^2 + b^2)}}{\sqrt{(1 + \alpha^2)}}, \qquad p_2 = \frac{\alpha ae + \sqrt{(a^2\, \alpha^2 + b^2}}{\sqrt{(1 + \alpha^2)}};$$

therefore

$$p_1 p_2 = \frac{a^2\, \alpha^2 + b^2 - a^2\, e^2\, \alpha^2}{1 + \alpha^2} = \frac{a^2\, \alpha^2 + b^2 - a^2\, \alpha^2 + b^2\, \alpha^2}{1 + \alpha^2} = b^2.$$

3. To find the locus of the extremity of the perpendiculars from the foci on the tangent.

The equation to the tangent is

$$y = ax + \sqrt{(a^2 a^2 + b^2)}.$$

The equation to the perpendicular on it from the focus is

$$y = -\frac{1}{a}(x - ae),$$

or

$$ay = -x + \sqrt{(a^2 - b^2)}.$$

Transposing, squaring, and adding, we have

$$(1 + a^2)(x^2 + y^2) = (1 + a^2) a^2, \text{ or } x^2 + y^2 = a^2.$$

We shall not proceed to shew how by the same means we can find the locus of the perpendicular from the centre on the tangent, or how to prove that $AT \,.\, at = b^2$ (*Ham. Con. Sec.* p. 108), as our readers can easily do so for themselves.

The analogous properties of the hyperbola may be proved in the same way by the use of the equation

[11] $$y = ax + \sqrt{(a^2 a^2 - b^2)},$$

and those of the parabola from the equation

$$y = ax + \frac{m}{a}.$$

But we need do no more than indicate this, as the method is the same as in the ellipse.

A. S.

ON GENERAL DIFFERENTIATION.

The idea of differential coefficients with general indices is not modern, for it occurred to Leibnitz, who has expressed it in his correspondence with Jean Bernouilli. Euler has written a few pages on this subject, which Lacroix has copied into his large work on the differential calculus. Formulæ for expressing the general differential coefficients of functions by means of definite integrals, have been given by Laplace (*Théorie des Probabilités*, p. 85, 3rd edit.), by Fourier (*Théorie de la Chaleur*, p. 561), and by Mr. Murphy (*Cambridge Phil. Trans.*, vol. v.) But it appears that the only person who has attempted to reduce the subject to a system, is M. Joseph Liouville; three memoirs by whom,—one on the principles of the calculus, and two on applications of it,—are inserted in the 13th volume of the *Journal de l'Ecole Polytechnique*, for the year 1832. Professor Peacock, in his valuable and interesting Report on certain branches of Analysis, which

forms a part of the *Report of the British Association for* 1833, has spoken of M. Liouville's system as erroneous in many essential points, and has given a sketch of one very different. But after referring to M. Liouville's memoirs, and bestowing considerable attention on the subject, we have come to a contrary opinion, at least with respect to his conclusions, which are the same for the most part as will be found in this article. Some points in his theory we admit to be objectionable, and these we have altered.

2. The transition from differential coefficients whose indices are positive integers, to those whose indices are any whatever, should be made in the same manner as the transition in algebra, from symbols of quantity with positive integral indices to those with general indices. Before we can prove any equations involving $\frac{d^a u}{dx^a}$, where a is general, we must affix a meaning to that expression, which can only be done by making some definition or assumption respecting it. [12] The assumption ought to be such that our results may coincide with the known results of the differential calculus when a becomes a positive integer. We shall therefore assume that the following equations, proved for differential coefficients with positive integral indices, hold true for differential cofficients with general indices:

$$\frac{d^a(u+v)}{dx^a} = \frac{d^a u}{dx^a} + \frac{d^a v}{dx^a} \dots\dots\dots\dots (A),$$

$$\frac{d^a}{dx^a} \cdot \frac{d^\beta}{dx^\beta} u = \frac{d^{a+\beta} u}{dx^{a+\beta}} \dots\dots\dots\dots\dots (B),$$

$$\frac{d^a}{dx^a} \cdot \frac{d^\beta u}{dy^\beta} = \frac{d^\beta}{dy^\beta} \cdot \frac{d^a u}{dx^a} \dots\dots\dots\dots (C).$$

3. From equation (A) it follows that if a be a constant,

$$\frac{d^a . au}{dx^a} = a \frac{d^a u}{dx^a}.$$

When a is a positive integer, this is very easily proved by making $v = u$, $2u$, $3u$, ... to $(a-1)u$, in succession. Next, let $a = \frac{p}{q}$, p and q being positive integers. Then, by the former case,

$$q \frac{d^a . \frac{p}{q} u}{dx^a} = \frac{d^a . q . \frac{p}{q} u}{dx^a} = \frac{d^a . pu}{dx^a} = p \frac{d^a u}{dx^a};$$

therefore

$$\frac{d^\alpha . \frac{p}{q} u}{dx^\alpha} = \frac{p}{q} \frac{d^\alpha u}{dx^\alpha}.$$

The proposition is also easily proved for a negative constant by assuming, in equation (A), $v = -u$. It seems that it cannot be proved when a^c_a is not a real quantity, but we shall extend the proposition to this case by assumption.

4. From equation (B) it may be easily deduced that

$$\left\{\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}}\right\}^q u = \frac{d^p u}{dx^p},$$

that is, that the operation denoted by $\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}}$ is such, that being performed q times in succession upon u, the result will be $\frac{d^p u}{dx^p}$. The same equation also enables us to interpret the meaning of $\frac{d^{-n}u}{dx^{-n}}$, n being an integer, for by making $a = -n$, and $\beta = n$, in equation (B), we have

[13]
$$\frac{d^{-n}}{dx^{-n}} \cdot \frac{d^n u}{dx^n} = \frac{d^0 u}{dx^0} = u;$$

whence it follows that $\frac{d^{-n}}{dx^{-n}}$ is the inverse of the operation $\frac{d^n}{dx^n}$. But we know that the inverse of $\frac{d^n}{dx^n}$ is the n^{th} integral with respect to x, therefore $\frac{d^{-n}}{dx^{-n}}$ denotes the n^{th} integral with respect to x.

5. The most important conclusion to be deduced from (C) is obtained by supposing $\beta = -1$, whence

$$\frac{d^\alpha}{dx^\alpha} \int u\, dy = \int \frac{d^\alpha u}{dx^\alpha}\, dy.$$

In deducing this formula the integral has been supposed indefinite: but it is easy to see that differentiation, under the sign of integration, is also allowable when the integral is taken between limits, provided that neither of the limiting values of y involve the parameter x.

6. Since, as has been shewn in § (4), general differentiation includes integration as a particular case, and since the com-

plete expression of an integral involves arbitrary constants, it follows that the complete expression of a general differential coefficient must involve arbitrary constants. We may express this by saying that the general value of $\dfrac{d^{\alpha}0}{dx^{\alpha}}$ is not 0. It is evident also, that we may introduce the proper arbitrary constants into any expression involving $\dfrac{d^{\alpha}u}{dx^{\alpha}}$, by adding to this quantity $\dfrac{d^{\alpha}0}{dx^{\alpha}}$, which we shall call with Liouville the *complementary function*. We may therefore neglect the complementary function in investigating general formulæ. Its form will be investigated hereafter.

7. We proceed to investigate the values of the general differential coefficients of various simple functions, and shall begin with ε^{nx}, because the result is easiest to be obtained, and may be made the foundation of all the rest of the calculus.

Put
$$y = \varepsilon^{nx},$$
then y satisfies the equation
$$\frac{dy}{dx} - ny = 0.$$

Hence, by equation (A) and § (3),
$$\frac{d^{\alpha}}{dx^{\alpha}}\frac{dy}{dx} - n\frac{d^{\alpha}y}{dx^{\alpha}} = 0.$$

By equation (B)
$$\frac{d^{\alpha}}{dx^{\alpha}}\frac{dy}{dx} = \frac{d^{\alpha+1}y}{dx^{\alpha+1}} = \frac{d}{dx}\frac{d^{\alpha}y}{dx^{\alpha}},$$

therefore
$$\frac{d}{dx}\cdot\frac{d^{\alpha}y}{dx^{\alpha}} - n\frac{d^{\alpha}y}{dx^{\alpha}} = 0, \qquad [14]$$

whence
$$\frac{d^{\alpha}y}{dx^{\alpha}} = C\varepsilon^{nx}.$$

Let $\alpha = \dfrac{p}{q}$, then, by § (4),
$$C^{q}\varepsilon^{nx} = \frac{d^{p}\varepsilon^{nx}}{dx^{p}} = n^{p}\varepsilon^{nx},$$

whether p be positive or negative. Hence $C = n^{\frac{p}{q}}$, and
$$\frac{d^{\alpha}\varepsilon^{nx}}{dx^{\alpha}} = n^{\alpha}\varepsilon^{nx} \quad \ldots\ldots\ldots\ldots\ldots \quad (D).$$

8. Since, if x be positive,

$$\frac{1}{x} = \int_0^\infty \varepsilon^{-\gamma x} d\gamma,$$

$$\frac{d^a}{dx^a}\frac{1}{x} = \int_0^\infty \varepsilon^{-\gamma x}(-\gamma)^a\, d\gamma.$$

Let $\gamma x = \theta$, then the integral is changed to

$$(-1)^a \cdot \frac{1}{x^{1+a}} \cdot \int_0^\infty \varepsilon^{-\theta}\theta^a d\theta.$$

If we designate, as Legendre has done, the definite integral

$$\int_0^\infty \varepsilon^{-\theta}\theta^{a-1} d\theta, \text{ by } \Gamma(a),$$

we have then $\dfrac{d^a}{dx^a}\dfrac{1}{x} = (-1)^a \cdot \Gamma(1+a) \cdot \dfrac{1}{x^{1+a}} \ldots\ldots\ldots (E).$

We have supposed x positive, but since the formula is not altered after changing x into $-x$ and reducing, it holds whether x be positive or negative.

9. To find the value of $\dfrac{d^a}{dx^a} \cdot \dfrac{1}{x^n}$, we proceed as follows.

Supposing x positive, and making $\gamma x = \theta$ in the definite integral

$$\int_0^\infty \varepsilon^{-\gamma x}\gamma^{n-1} d\gamma,$$

we find $\displaystyle\int_0^\infty \varepsilon^{-\gamma x}\gamma^{n-1} d\gamma = \frac{1}{x^n} \cdot \int_0^\infty \varepsilon^{-\theta}\theta^{n-1} d\theta = \Gamma(n) \cdot \frac{1}{x^n}.$

Therefore $$\frac{1}{x^n} = \frac{\int_0^\infty \varepsilon^{-\gamma x}\gamma^{n-1} d\gamma}{\Gamma(n)},$$

$$\frac{d^a}{dx^a}\frac{1}{x^n} = \frac{\int_0^\infty \varepsilon^{-\gamma x}(-\gamma)^a \gamma^{n-1}\, d\gamma}{\Gamma(n)};$$

[15] and by making $\gamma x = \theta$,

$$\frac{d^a}{dx^a}\frac{1}{x^n} = (-1)^a \cdot \frac{\Gamma(n+a)}{\Gamma(n)} \cdot \frac{1}{x^{n+a}} \ldots\ldots\ldots\ldots (F).$$

The remark at the end of the last section applies equally to this expression.

10. We shall digress to prove a few of the most important properties of the definite integral $\Gamma(n)$. It is the second of the Eulerian integrals, as Legendre has called them, from

their having been first treated of by Euler. It may be put into other forms besides $\int_0^\infty \varepsilon^{-\theta}\theta^{n-1}d\theta$, for by making $\varepsilon^{-\theta} = x$, this becomes

$$\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx:$$

again, by making $\theta^n = z$, the former expression becomes

$$\frac{1}{n}\int_0^\infty \varepsilon^{-z^{\frac{1}{n}}}\, dz.$$

11. The first of the properties of the integral in question is, that if n be positive,

$$\Gamma(1+n) = n\Gamma(n) \ldots\ldots\ldots\ldots (G).$$

For, integrating by parts,

$$\int \varepsilon^{-\theta}\theta^n d\theta = -\varepsilon^{-\theta}\theta^n + n\int \varepsilon^{-\theta}\theta^{n-1}d\theta.$$

Now the integrated part vanishes when $\theta = \infty$, and also when $\theta = 0$, provided n be positive; hence, in this case,

$$\Gamma(1+n) = n\Gamma(n).$$

If $n = 0$, the integrated part becomes -1 when $\theta = 0$; and if n be negative, it becomes ∞; so that in neither of these cases the equation (G) is true.

It follows from equation (G) that if r be any integer less than $1 + n$,

$$\frac{\Gamma(1+n)}{\Gamma(1+n-r)} = n(n-1)(n-2)\ldots(n-r+1)\ldots (H).$$

If n be a positive integer we may make $r = n$; whence, observing that $\Gamma(1) = \int_0^\infty \varepsilon^{-\theta}d\theta = 1$, we find

$$\Gamma(1+n) = n(n-1)(n-2)\ldots 2.1 \ldots\ldots (I).$$

12. It is desirable to examine what $\Gamma(n)$ becomes when n is 0 or negative. For this purpose, let it be observed that the integral

$$\int_0^\infty \varepsilon^{-\theta}\theta^{n-1}\,d\theta$$

is equivalent to the infinite series

$$\varepsilon^{-d\theta}(d\theta)^{n-1}d\theta + \varepsilon^{-2d\theta}(2d\theta)^{n-1}d\theta + \varepsilon^{-3d\theta}(3d\theta)^{n-1}d\theta + \ldots\ldots$$

or $\quad \Gamma(n) = \left\{\varepsilon^{-d\theta}.1^{n-1} + \varepsilon^{-2d\theta}.2^{n-1} + \varepsilon^{-3d\theta}.3^{n-1} + \ldots\right\}(d\theta)^n.$

Now as long as n is positive, the sum of the series within the brackets is infinite, but it is multiplied by the infinitely small quantity $(d\theta)^n$; there is no reason therefore why the value of the expression should not be finite. When $n = 0$, the [16]

above expression for $\Gamma(n)$ becomes

$$-\log(1-\varepsilon^{-d\theta}) = -\log 0 = \infty.$$

When n is negative, the series within the brackets becomes finite, but it is multiplied by the infinite quantity $(d\theta)^n$; therefore $\Gamma(n)$ is infinite when n is negative. It may be remarked that though $\Gamma(0)$ and $\Gamma(-n)$ are both infinite, $\dfrac{\Gamma(0)}{\Gamma(-n)}$ is 0, because $(d\theta)^n \log(1-\varepsilon^{-d\theta})$ is 0, however small n may be.

13. To prove that

$$\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} = \left(\frac{m}{n}\right) \ldots\ldots\ldots\ldots (K),$$

where $\left(\dfrac{m}{n}\right)$ denotes $\displaystyle\int_0^1 (1-x)^{m-1}x^{n-1}dx$, which is called the first Eulerian integral.*

We have $\displaystyle \Gamma(m).\Gamma(n) = \int_0^\infty \varepsilon^{-x} x^{m-1}\,dx \,.\int_0^\infty \varepsilon^{-y} y^{n-1}\,dy$

$$= \int_0^\infty\int_0^\infty \varepsilon^{-x-y} x^{m-1} y^{n-1}\,dx\,dy:$$

change y into xy, and dy into xdy, then the last expression becomes

$$\int_0^\infty\int_0^\infty \varepsilon^{-x(1+y)} x^{m+n-1} y^{n-1}\,dx\,dy.$$

Change x into $\dfrac{x}{1+y}$, and dx into $\dfrac{dx}{1+y}$, then this becomes

$$\int_0^\infty\int_0^\infty \varepsilon^{-x} x^{m+n-1} \frac{y^{n-1}}{(1+y)^{m+n}}\,dx\,dy$$

$$= \int_0^\infty \varepsilon^{-x} x^{m+n-1}\,dx \,.\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy.$$

The first factor is $\Gamma(m+n)$; the second is to be transformed as follows. Assume $1+y=\dfrac{1}{1-z}$, then $dy=\dfrac{dz}{(1-z)^2}$, and $y=\dfrac{z}{1-z}$; when $y=0$, $z=0$, and when $y=\infty$, $z=1$.

Hence $\displaystyle \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^1 (1-z)^{m-1} z^{n-1}\,dz = \left(\frac{m}{n}\right);$

and the proposition is manifest.

Supposing m and n, which are in general fractions, to be reduced to a common denominator, and to be equal to $\dfrac{q}{r}$, $\dfrac{p}{r}$,

---

* See p. [94].

and changing z into x^r, the integral

$$\int_0^1 (1-z)^{m-1} z^{n-1}\,dz \text{ becomes } r\int_0^1 \frac{x^{p-1}\,dx}{(1-x^r)^{1-\frac{q}{r}}},$$

which is the form adopted by several writers. A good many of its properties are proved in Hymers's *Integral Calculus.*

14. Let $n = 1 - m$, then $\Gamma(m+n)$ becomes 1, and [17] the equation just proved becomes

$$\Gamma(m)\,\Gamma(1-m) = \int_0^1 (1-z)^{-1+m} z^{-m}\,dz;$$

the value of which integral is $\dfrac{\pi}{\sin m\pi}$, if m be between 0 and 1.

The following is a new method of finding it:

Let $z = (\sin\theta)^2$, then $1 - z = (\cos\theta)^2$, $dz = 2\sin\theta\cos\theta\,d\theta$, when $z = 0$, $\theta = 0$, and when $z = 1$, $\theta = \frac{1}{2}\pi$,

Hence $$\int_0^1 (1-z)^{-1+m} z^{-m}\,dz = 2\int_0^{\frac{1}{2}\pi} (\tan\theta)^{1-2m}\,d\theta.$$

If we put for $\tan\theta$ its value $\dfrac{1}{\sqrt{(-1)}}\,\dfrac{1-\varepsilon^{-\sqrt{(-1)}2\theta}}{1+\varepsilon^{-\sqrt{(-1)}2\theta}}$, it is evident that $\left(\dfrac{1-\varepsilon^{-\sqrt{(-1)}2\theta}}{1+\varepsilon^{-\sqrt{(-1)}2\theta}}\right)^{1-2m}$ may be expanded in the form

$$\begin{aligned} &1 + A_1\varepsilon^{-\sqrt{(-1)}2\theta} + A_2\varepsilon^{-\sqrt{(-1)}4\theta} + \dots \\ &= 1 + A_1\cos 2\theta + A_2\cos 4\theta + \dots \\ &\quad - \sqrt{(-1)}\,(A_1\sin 2\theta + A_2\sin 4\theta + \dots). \end{aligned}$$

Also, $$\{\sqrt{(-1)}\}^{1-2m} = \cos(1-2m)\tfrac{1}{2}\pi + \sqrt{(-1)}\sin(1-2m)\tfrac{1}{2}\pi = \sin m\pi + \sqrt{(-1)}\cos m\pi.$$

Substituting the series, multiplying by the above value of $\{\sqrt{(-1)}\}^{1-2m}$, and equating real parts, we find

$$\sin m\pi \int_0^{\frac{1}{2}\pi} (\tan\theta)^{1-2m}\,d\theta = \int_0^{\frac{1}{2}\pi} (1 + A_1\cos 2\theta + A_2\cos 4\theta + \dots)\,d\theta = \tfrac{1}{2}\pi,$$

since the periodic terms vanish at each limit. Hence

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi} \quad \dots\dots\dots\dots (L).$$

It is necessary that m should be less than 1, otherwise $(\tan\theta)^{1-2m}\,d\theta$ would not be infinitely small when θ is so.

15. In equation (L), put $m = \frac{1}{2}$, therefore $\{\Gamma(\frac{1}{2})\}^2 = \pi$, and

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Hence, by equation (G), § 11, we find

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi},\quad \Gamma\left(\frac{5}{2}\right) = \frac{3.1}{2^2}\sqrt{\pi}, \ldots\ldots$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2^n}\sqrt{\pi}.$$

[18] 16. We now return to our subject of general differentiation. Our readers will find no difficulty in applying equation (H) to obtain from formula (F) the known values of $\dfrac{d^a}{dx^a}\dfrac{1}{x^n}$ when a is an integer, positive or negative, provided that $n + a$ be positive. But that formula must not be extended to cases where n or $n + a$ is negative. For it depends upon the equation

$$\frac{1}{x^r} = \frac{\int_0^\infty \varepsilon^{-\gamma x}\gamma^{r-1}d\gamma}{\Gamma(r)};$$

which represents the development of $\dfrac{1}{x^r}$ in a series of exponentials; and the infinite value of $\Gamma(r)$ when r is negative shews that a positive power of x cannot be developed in such a series. We cannot therefore rely upon results which are obtained by supposing this equation true for negative values of r. But, without supposing formula (F) to hold when the index of x on either side is positive, we may deduce from it formulæ to suit such cases.

17. To find $\dfrac{d^a x^n}{dx^a}$ when n is positive and $n - a$ negative.

By formula (F) $\dfrac{d^a}{dx^a}\dfrac{1}{x^n} = (-1)^a.\dfrac{\Gamma(n+a)}{\Gamma(n)}.\dfrac{1}{x^{n+a}}$;

where we suppose, for convenience, n to be between 0 and 1. Integrate both sides p times, p being any number less than $n + a$; then the first side becomes

$$\frac{d^a}{dx^a}\int^p \frac{1}{x^n}dx^p = \frac{1}{(1-n)(2-n)\ldots\ldots(p-n)}.\frac{d^a}{dx^a}x^{p-n}$$
$$= \frac{\Gamma(1-n)}{\Gamma(1+p-n)}.\frac{d^a}{dx^a}.x^{p-n};$$

and the second side becomes

$$(-1)^a.\frac{\Gamma(n+a)}{\Gamma(n)}.(-1)^p.\frac{1}{(n+a-1)(n+a-2)\ldots(n+a-p)}.\frac{1}{x^{n+a-p}}$$
$$= (-1)^{a+p}\frac{\Gamma(n+a-p)}{\Gamma(n)}\frac{1}{x^{n+a-p}};$$

wherefore

$$\frac{d^a}{dx^a} x^{p-n} = (-1)^{a+p} \cdot \frac{\Gamma(1+p-n)\,\Gamma(a-p+n)}{\Gamma(n)\,\Gamma(1-n)} \cdot \frac{1}{x^{a-p+n}}:$$

but since n is between 0 and 1, by equation (L),

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} = (-1)^{p-1} \frac{\pi}{\sin(p-n)\pi}.$$

Making this substitution, and changing $p - n$ into n,

$$\frac{d^a x^n}{dx^a} = (-1)^{a+1} \frac{\sin n\pi}{\pi} \cdot \Gamma(1+n) \cdot \Gamma(a-n) \cdot \frac{1}{x^{a-n}} \;\ldots\ldots\; (M).$$

18. From this formula we may immediately deduce one for the contrary case, namely that where the index of x is negative before differentiation and positive afterwards, the index of differentiation being negative. For, affecting both sides of the last equation with $\frac{d^{-a}}{dx^{-a}}$, and dividing by the constants,

$$\frac{d^{-a}}{dx^{-a}} \frac{1}{x^{a-n}} = (-1)^{a+1} \cdot \frac{\pi}{\sin n\pi \cdot \Gamma(1+n)\,\Gamma(a-n)} \cdot x^n;$$

and changing $a - n$ into n, and therefore n into $a - n$,

$$\frac{d^{-a}}{dx^{-a}} \frac{1}{x^n} = (-1)^{a+1} \cdot \frac{\pi}{\sin(a-n)\pi \cdot \Gamma(1+a-n)\,\Gamma(n)} \cdot x^{a+n} \ldots (N).$$

19. It remains to find a formula for the case where the index of x is positive both before and after differentiation. For this purpose suppose $a - n$ to be between 0 and 1 in formula (M), and integrate p times, then the first side becomes

$$\frac{1}{(n+1)(n+2)\ldots(n+p)} \cdot \frac{d^a x^{n+p}}{dx^a} = \frac{\Gamma(1+n)}{\Gamma(1+n+p)} \cdot \frac{d^a x^{n+p}}{dx^a}:$$

and on the second side the index of x will become $p + n - a$, and it will be multiplied by

$$\frac{1}{(1-a+n)(2-a+n)\ldots.(p-a+n)} = \frac{\Gamma(1-a+n)}{\Gamma(1+n+p-a)}.$$

Therefore $\dfrac{d^a x^{n+p}}{dx^a} = (-1)^{a+1} \cdot \dfrac{\sin n\pi}{\pi} \times$

$$\Gamma(a-n) \cdot \Gamma(1-a+n) \cdot \frac{\Gamma(1+n+p)}{\Gamma(1+n+p-a)} \cdot x^{p+n-a}.$$

Now [19]

$$\Gamma(a-n) \cdot \Gamma(1-a+n) = \frac{\pi}{\sin(a-n)\pi} = \frac{\pi}{(-1)^{p+1} \sin(n+p-a)\pi},$$

and $\sin n\pi = (-1)^p \sin(n+p)\pi$. Substituting, and changing $n+p$ into n, we find

$$\frac{d^a x^n}{dx^a} = (-1)^a \cdot \frac{\sin n\pi}{\sin(n-a)\pi} \cdot \frac{\Gamma(1+n)}{\Gamma(1+n-a)} \cdot x^{n-a} \;\ldots\ldots\; (O).$$

This formula may be easily shewn, by the same process as in § 18, to be true when a is negative.

The factor in the last expression, $(-1)^a \dfrac{\sin n\pi}{\sin(n-a)\pi}$, is remarkable for becoming equal to unity whenever a is an integer, while it admits of any value between $+\infty$ and $-\infty$ when a is fractional.

[20] 20. The most important features in the formulæ just investigated are the sines of multiples of the semicircumference. From formulæ (M) and (O) it follows that the differentials to fractional indices of positive integral powers of x are nothing; and from formulæ (N) and (O), that when the index of x is fractional or negative before differentiation, and a positive integer after it, the differential coefficient is infinite. It is true that in the investigations we excluded the cases of the index of x being integral before or after differentiation, but that was only for convenience. In order to find the results in those cases, it would be necessary, where we supposed the index of x to be between 0 and -1, instead of that to suppose it equal to -1, and by considering that $\dfrac{x^0}{0}$, as a value of $\displaystyle\int \frac{dx}{x}$, is not false, but only differs from the value $\log x$ by an infinite arbitrary constant, and that our object here is to get the value of $\dfrac{d^a x^n}{dx^a}$ in the form Mx^{n-a}; we shall see that our formulæ (M), (N), (O), give the right results in the above-mentioned cases. The only restriction upon the generality of the formulæ is that we must not make quantities under the sign Γ negative or nothing.

21. It is desirable to obtain a formula for $\dfrac{d^a x^n}{dx^a}$ which shall be generally true whatever be the values of a and n. This may be done, though not in terms of the function Γ.

Assume
$$\frac{d^a x^n}{dx^a} = Mx^{n-a};$$

and take the $(n-a)^{\text{th}}$ differential coefficient of both sides, therefore
$$\frac{d^n x^n}{dx^n} = M\frac{d^{n-a} x^{n-a}}{dx^{n-a}}.$$

The value of $\frac{d^n x^n}{dx^n}$ is independent of x; let it be represented by $P(n)$, then we have

$$M = \frac{P(n)}{P(n-a)},$$

and
$$\frac{d^a x^n}{dx^a} = \frac{P(n)}{P(n-a)} \cdot x^{n-a} \quad \ldots\ldots\ldots \quad (P).$$

It appears from formulæ (N) and (O), that the value of $P(n)$ is infinite in every case except when n is a positive integer, in which case it becomes $1.2.3 \ldots . n$. It will in all cases possess the property

$$P(n) = n\,P(n-1).$$

22. We are now enabled to assign the form of the [21] complementary function. The quantity which is to be added to the a^{th} differential coefficient of any function to render it complete, must evidently be one of which the $(-a)^{\text{th}}$ differential coefficient is nothing. But we have seen in § 20, that the fractional differential coefficient of a power of x vanishes when the index of that power is a positive integer, and in no other case; consequently the form of the complementary function is
$$C_0 + C_1 x + C_2 x^2 + \ldots .$$
the number of terms being indefinite when the index of differentiation is a fraction.

ON A PROPERTY OF THE TRIANGLE.

The following property of a triangle is remarkable not only for the curious relation between certain lines, but for its leading readily to two elegant theorems regarding the radii of the circles which touch the three sides of a triangle.

Let x, y, z be the perpendiculars from any point on the sides of a triangle, p, q, r the perpendiculars respectively parallel to them through the angles. Then

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

If we join the point from which the perpendiculars are drawn with the three angles, the whole triangle will be divided into three triangles, which have the sides of the triangles as bases, and the lines x, y, z as their vertical

heights. Let a, b, c be the sides of the triangle, then $\frac{ax}{2}$, $\frac{by}{2}$, $\frac{cz}{2}$ will be the areas of the three parts; and $\frac{ax+by+cz}{2}$ = area of whole triangle.

Also area of whole triangle $= \frac{ap}{2} = \frac{bq}{2} = \frac{cr}{2}$.

Dividing the corresponding terms by these quantities we get

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

This relation between the lines is evidently that of the coordinates of a plane which cuts the axes at points whose distances from the origin are p, q, r.

[22] If the point were outside of the triangle, we should have to subtract one of the terms, such as cz, so that the resulting equation would be

$$\frac{x}{p} + \frac{y}{q} - \frac{z}{r} = 1.$$

Now let ρ be the radius of the inscribed circle, then taking the centre of this circle as the given point, we have $x = y = z = \rho$, and consequently

$$\frac{1}{\rho} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Again, let ρ_1, ρ_2, ρ_3 be the radii of the circles which touch one of the sides of the triangle externally, and the other two internally; then we shall have, by similar reasoning,

$$\frac{1}{\rho_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r}$$

$$\frac{1}{\rho_2} = \frac{1}{p} - \frac{1}{q} + \frac{1}{r}$$

$$\frac{1}{\rho_3} = -\frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Adding these equations together, we get

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{\rho}.$$

It is obvious that similar theorems hold good for any tetrahedron, but it is needless to do more than indicate them.

ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

The following method of integrating linear differential equations deserves attention, not only as leading readily to the solution of these equations, but also as placing their theory in a clear light, and pointing out the cause of the success of the method usually employed.

M. Brisson appears to have been the first person who applied the principle of the separation of the signs of operation from those of quantity to the solution of differential equations. This he did in two memoirs of the dates of 1821 and 1823, but we have not been fortunate enough to meet with them (if indeed they have been published), and our knowledge of them is derived from a casual notice in [23]
a memoir of Cauchy on the same subject, in his *Exercices*, vol. ii. p. 159. This last author seems to have pursued a different course from Brisson; and as it does not appear to be the best for putting the subject in a clear light, we have taken the liberty of deviating very considerably from his method, and in so doing we have probably approached nearer to that of Brisson.

If we take the general linear equation with constant coefficients,

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \ldots + R\frac{dy}{dx} + Sy = X,$$

when X is any function of x, and separate the signs of operation from those of quantity, it becomes

$$\left(\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \ldots + R\frac{d}{dx} + S\right)y = X.$$

The quantity within the brackets involving only constants, and the signs of operation may be considered as one operation performed on y, and it may be represented by

$$f\left(\frac{d}{dx}\right)y = X.$$

Here y is given at once explicitly if we are able to perform the inverse operation of $f\left(\frac{d}{dx}\right)$. For if we represent the

inverse operation by the usual symbol $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$, and perform that operation on both sides, we get

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1} \cdot f\left(\frac{d}{dx}\right) y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X,$$

or,
$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X.$$

It is plain that in its general form we cannot easily perform the inverse operation $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$; but if we begin with a simple case we shall be easily led to a means of effecting it.

Let us take the equation

$$y + \frac{dy}{dx} = ax^n,$$

or,
$$\left(1 + \frac{d}{dx}\right) y = ax^n.$$

Now the inverse operation of $\left(1 + \frac{d}{dx}\right)$ is $\left(1 + \frac{d}{dx}\right)^{-1}$. Therefore

$$y = \left(1 + \frac{d}{dx}\right)^{-1} ax^n.$$

[24] But as in integration there must be added an arbitrary constant which vanishes by differentiation, so here we must add a function which will vanish when the operation $\left(1 + \frac{d}{dx}\right)$ is performed on it. This complementary function may be found from that condition, but the following more direct method is perhaps preferable. Since the result of the operation $1 + \frac{d}{dx}$ on the function is 0, we may put the value of y under the form

$$y = \left(1 + \frac{d}{dx}\right)^{-1} ax^n + \left(1 + \frac{d}{dx}\right)^{-1} 0.$$

Now if we treat the symbols of operation as if they were symbols of quantity, we have

$$\left(1 + \frac{d}{dx}\right)^{-1} 0 = \frac{d^{-1}}{dx^{-1}} \left(1 + \frac{d^{-1}}{dx^{-1}}\right)^{-1} 0.$$

But $\frac{d^{-1}}{dx^{-1}}$ is the same as $\int dx$. Hence

$$\left(1 + \frac{d}{dx}\right)^{-1} 0 = \left(1 + \frac{d^{-1}}{dx^{-1}}\right)^{-1} C,$$

(C being the arbitrary constant arising from the integration)

$$= \left(1 - \frac{d^{-1}}{dx^{-1}} + \frac{d^{-2}}{dx^{-1}} - \ldots\right) C;$$

or, performing the operations indicated,

$$= C\left(1 - x + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \ldots\right) = C\epsilon^{-x}.$$

Hence
$$y = \left(1 + \frac{d}{dx}\right)^{-1} ax^n + C\epsilon^{-x}.$$

Now, expanding the first term,

$$y = \left(1 - \frac{d}{dx} + \frac{d^2}{dx^2} - \frac{d^3}{dx^3} + \ldots\right) ax^n + C\epsilon^{-x}.$$

Therefore

$$y = a\,(x^n - nx^{n-1} + n.n - 1x^{n-2} - \ldots) + C\epsilon^{-x}.$$

As the operation $\left(1 + \frac{d}{dx}\right)^{-1}$ frequently occurs in these equations it is convenient to recollect that we must always add the function $C\epsilon^{-x}$. And in the same way it would be seen if the operation be $\left(a + \frac{d}{dx}\right)^{-1}$, the complementary function is $C\epsilon^{-ax}$, and similarly for all binomial symbols of operation of this kind.

Equations of the first degree, when the coefficients of y [25] and $\frac{dy}{dx}$ are functions of x, are easily reduced to this case by a change of the independent variable. Let us take as an example the equation

$$\frac{dy}{dx} + \frac{ny}{\sqrt{(1+x^2)}} = a,$$

or
$$\left(1 + \frac{\sqrt{(1+x^2)}}{n}\frac{d}{dx}\right) y = \frac{a\sqrt{(1+x^2)}}{n}.$$

Let $\frac{ndx}{\sqrt{(1+x^2)}} = dt$, therefore $\frac{t}{n} = \log\{x + \sqrt{(1+x^2)}\}$;

whence
$$\sqrt{(1+x^2)} = \tfrac{1}{2}\left(\epsilon^{\frac{t}{n}} + \epsilon^{-\frac{t}{n}}\right),$$

and the equation becomes

$$\left(1 + \frac{d}{dt}\right) y = \frac{a}{2n}\left(\epsilon^{\frac{t}{n}} + \epsilon^{-\frac{t}{n}}\right);$$

therefore $$y=\left(1+\frac{d}{dt}\right)^{-1}\frac{a}{2n}\left(\varepsilon^{\frac{t}{n}}+\varepsilon^{-\frac{t}{n}}\right)+c\varepsilon^{-t};$$

or, expanding the first term,

$$y=\frac{a}{2n}\left(1-\frac{d}{dt}+\frac{d^2}{dt^2}-\&c.\right)\left(\varepsilon^{\frac{t}{n}}+\varepsilon^{-\frac{t}{n}}\right)+c\varepsilon^{-t}$$

$$=\frac{a}{2n}\left(1-\frac{1}{n}+\frac{1}{n^2}-\&c.\right)\varepsilon^{\frac{t}{n}}$$

$$+\frac{a}{2n}\left(1+\frac{1}{n}+\frac{1}{n^2}-\&c.\right)\varepsilon^{-\frac{t}{n}}+c\varepsilon^{-t};$$

therefore $$y=\frac{a}{2(n+1)}\varepsilon^{\frac{t}{n}}+\frac{a}{2(n-1)}\varepsilon^{-\frac{t}{n}}+c\varepsilon^{-t},$$

or, substituting for t its value in terms of x,

$$y=\frac{a}{2(n+1)}\{\sqrt{(1+x^2)}+x\}+\frac{a}{2(n-1)}\{\sqrt{(1+x^2)}-x\}$$
$$+c\{\sqrt{(1+x^2)}-x\}^n.$$

It is needless to multiply examples, as the principle of the method in the case of equations of the first order is sufficiently obvious from those given. But we will proceed to prove a theorem which is very useful, particularly in equations of the higher orders. The theorem is, that

$$\left(\frac{d}{dx}\pm a\right)^n X=\varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n\varepsilon^{\pm ax}X.$$

For, if we expand the first side, we have

$$\left(\frac{d}{dx}\pm a\right)^n X=\left(\frac{d^n}{dx^n}\pm na\frac{d^{n-1}}{dx^{n-1}}+\frac{n.n-1}{1.2}a^2\frac{d^{n-2}}{dx^{n-2}}\pm\&c.\right)X.$$

[26] Now, $$\pm a^p=\varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^p\varepsilon^{\pm ax},$$

so that the second side may be put under the form

$$\varepsilon^{\mp ax}\left(\frac{d^n}{dx^n}+n\frac{d^{n-1}}{dx^{n-1}}\cdot\frac{d'}{dx}+\frac{n.n-1}{1.2}\frac{d^{n-2}}{dx^{n-2}}\frac{d'^2}{dx^2}+\&c.\right)\varepsilon^{\pm ax}X,$$

(where the accented letters refer to $\varepsilon^{\pm ax}$, and the unaccented to X), and this is equivalent to

$$\varepsilon^{\mp ax}\left(\frac{d}{dx}+\frac{d'}{dx}\right)^n\varepsilon^{\pm ax}X=\varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n\varepsilon^{\pm ax}X,$$

by the theorem of *Leibnitz*.

When $X = \varepsilon^{mx}$, the proposition takes the form

$$\left(\frac{d}{dx} \pm a\right)^n \varepsilon^{mx} = (m \pm a)^n \varepsilon^{mx}.$$

By this theorem all operations of the nature of $\left(\frac{d}{dx} \pm a\right)^n$ are reduced to differentiation, or, as in the cases to which we have generally to apply it n is negative, to integration.

To return now to the general equation which we represented by

$$f\left(\frac{d}{dx}\right) y = X.*$$

The inverse operation of $f\left(\frac{d}{dx}\right)$ cannot easily be performed directly, but we conceive the operation $f\left(\frac{d}{dx}\right)$ to be made up by the combination of n binomial operations of the form of $\left(\frac{d}{dx} - a\right)$; and, by what we have shewn before, we can perform the inverse operation for each of these successively, and this will be equivalent to performing the whole inverse operation of $f\left(\frac{d}{dx}\right)$ at once. For, treating the operation $\frac{d}{dx}$ exactly as if it were a function of x of the same form, we can resolve it into factors, so that it becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right),$$

where a_1, a_2, a_3, &c. are the roots of the equation

$$f(z) = 0.$$

Hence the equation $f\left(\frac{d}{dx}\right) y = X$ becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = X.$$

Now, performing the inverse operation of $\left(\frac{d}{dx} - a_1\right)$, [27] we have

$$\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right), \&c. \left(\frac{d}{dx} - a_n\right) y = \left(\frac{d}{dx} - a_1\right)^{-1} X$$
$$= \varepsilon^{a_1 x} \int \varepsilon^{-a_1 x} X dx,$$

by the theorem prefixed, since in this case $n = -1$.

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* See vol. II. p. 114.

We should properly add a term $\left(\frac{d}{dx} - a_1\right)^{-1} 0 = c\,\epsilon^{a_1 x}$, but as we may suppose the arbitrary constant to be included in the sign of integration, we may leave out this term for the sake of brevity.

Again, performing the inverse operation of $\left(\frac{d}{dx} - a_2\right)$, we have

$$\left(\frac{d}{dx} - a_3\right), \text{\&c.} \left(\frac{d}{dx} - a_n\right) y = \left(\frac{d}{dx} - a_2\right)^{-1} \left(\epsilon^{a_1 x} \int \epsilon^{-a_1 x} X\, dx\right)$$
$$= \epsilon^{a_2 x} \int \epsilon^{(a_1 - a_2) x} \left(\int \epsilon^{-a_1 x} X dx\right) dx.$$

Integrating by parts, this becomes

$$\left(\frac{d}{dx} - a_3\right), \text{\&c.} \left(\frac{d}{dx} - a_n\right) y = \frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{a_1 - a_2} - \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{a_1 - a_2}$$
$$= \frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{a_1 - a_2} - \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{a_2 - a_1}.$$

Performing the inverse operation of $\left(\frac{d}{dx} - a_3\right)$, we have

$$\left(\frac{d}{dx} - a_4\right), \text{\&c.} \left(\frac{d}{dx} - a_n\right) y$$
$$= \left(\frac{d}{dx} - a_3\right)^{-1} \left\{\frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{a_1 - a_2} + \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{a_2 - a_1}\right\}$$
$$= \frac{\epsilon^{a_3 x} \int \epsilon^{(a_1 - a_3) x} \left(\int \epsilon^{-a_1 x} X dx\right) dx}{a_1 - a_2} + \frac{\epsilon^{a_3 x} \int \epsilon^{(a_2 - a_3) x} \left(\int \epsilon^{-a_2 x} X dx\right.}{a_2 - a_1}.$$

And integrating each of the terms separately by parts, we get, as before,

$$\frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{(a_1 - a_2)(a_1 - a_3)} - \frac{\epsilon^{a_3 x}\left(\int \epsilon^{-a_3 x} X dx\right)}{(a_1 - a_2)(a_1 - a_3)}$$
$$+ \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{(a_2 - a_1)(a_2 - a_3)} - \frac{\epsilon^{a_3 x}\left(\int \epsilon^{-a_3 x} X dx\right)}{(a_2 - a_1)(a_2 - a_3)}$$
$$= \frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{(a_1 - a_2)(a_1 - a_3)} + \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{(a_2 - a_1)(a_2 - a_3)} + \frac{\epsilon^{a_3 x}\left(\int \epsilon^{-a_3 x} X dx\right)}{(a_3 - a_1)(a_3 - a_2)},$$

and so on for every successive factor, so that at last

$$y = \frac{\epsilon^{a_1 x}\left(\int \epsilon^{-a_1 x} X dx\right)}{(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n)} + \frac{\epsilon^{a_2 x}\left(\int \epsilon^{-a_2 x} X dx\right)}{(a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n)} + \text{\&c.}$$
$$+ \frac{\epsilon^{a_n x}\left(\int \epsilon^{-a_n x} X dx\right)}{(a_n - a_1)(a_n - a_2)\ldots(a_n - a_{n-1})}.$$

We shall leave to the reader the application of the general method to particular cases, and shall proceed to shew how some equations, of an order higher than the first, may be conveniently solved without operating with each factor separately. [28]

For instance, if we take the example of the equation

$$y + \frac{d^2y}{dx^2} = a \cos mx,$$

or
$$\left(1 + \frac{d^2}{dx^2}\right) y = a \cos mx;$$

therefore
$$y = \left(1 + \frac{d^2}{dx^2}\right)^{-1} a \cos mx + \left(1 + \frac{d^2}{dx^2}\right)^{-1} 0.$$

Now,
$$\left(1 + \frac{d^2}{dx^2}\right)^{-1} 0 = \frac{d^{-2}}{dx^{-2}}\left(1 + \frac{d^{-2}}{dx^{-2}}\right) 0$$
$$= \left(1 + \frac{d^{-2}}{dx^{-2}}\right)(c_1 x + c_2)$$
$$= \left(1 - \frac{d^{-2}}{dx^{-2}} + \frac{d^{-4}}{dx^{-4}} - \&c.\right)(c_1 x + c_2)$$
$$= c_1\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.\ldots.5} - \&c.\right)$$
$$+ c_2\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.\right)$$
$$= c_1 \sin x + c_2 \cos x.$$

Also,
$$\left(1 + \frac{d^2}{dx^2}\right)^{-1} a \cos mx = a\left(1 - \frac{d^2}{dx^2} + \frac{d^4}{dx^4} - \&c.\right) \cos mx$$
$$= a\,(1 + m^2 + m^4 + \&c.) \cos mx = \frac{a}{1 - m^2} \cos mx;$$

whence
$$y = \frac{a}{1 - m^2} \cos mx + c_1 \sin x + c_2 \cos x.$$

Again, if we have the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2,$$

where the binomial factors of operation are equal, it may be put under the form

$$\left(\frac{d}{dx} - 2\right)^2 y = x^2,$$

whence
$$y = \left(\frac{d}{dx} - 2\right)^{-2} x^2 + \left(\frac{d}{dx} - 2\right)^{-2} 0.$$

Now $$\left(\frac{d}{dx}-2\right)^{-2}x^2=\left(2-\frac{d}{dx}\right)^{-2}x^2$$

$$=\left(2^{-2}+2.2^{-3}\frac{d}{dx}+3.2^{-4}\frac{d^2}{dx^2}+\&c.\right)x^2=\frac{x^2}{2^2}+\frac{4x}{2^3}+\frac{6}{2^4}.$$

[29] Also,

$$\left(\frac{d}{dx}-2\right)^{-2}0=\frac{d^{-2}}{dx^{-2}}\left(1-2\frac{d^{-1}}{dx^{-1}}\right)^{-2}0=\left(1-2\frac{d^{-1}}{dx^{-1}}\right)^{-2}(cx+c_1)$$

$$=\left(1+2.2\frac{d^{-1}}{dx^{-1}}+3.2^2\frac{d^{-2}}{dx^{-2}}+4.2^3\frac{d^{-3}}{dx^{-3}}+\&c.\right)(cx+c_1)$$

$$=c\left(x+2.\frac{2x^2}{1.2}+3.\frac{2^2x^3}{1.2.3}+\&c.\right)$$

$$+c_1\left(1+2.2x+3.2^2\frac{2^2}{1.2}+\&c.\right)$$

$$=cx\left(1+2x+\frac{2^2x^2}{1.2}+\frac{2^3x^3}{1.2.3}+\&c.\right)$$

$$+c_1\left(1+2x+\frac{2^2x^2}{1.2}+\&c.\right)+2c_1x\left(1+2x+\frac{2^2x^2}{1.2}+\&c.\right)$$

$$=(c_1+c_2x(\varepsilon^{2x},\text{ if }c+2c_1=c_2;$$

therefore we have

$$y=\frac{x^2}{2^2}+\frac{4x}{2^3}+\frac{6}{2^4}+(c_1+c_2x)\,\varepsilon^{2x}.$$

We might have omitted the latter part of this example, as it is easy to shew, in the usual manner, what is the form of the complementary function when the two factors are equal, but we preferred the method given, as shewing how we may arrive at the same result directly.

On looking back on the method pursued, it is easy to see the causes of some of the known peculiarities in the usual solution of linear differential equations with constant coefficients. In the first place, their solution is attended with greater facility than that of other differential equations, because in fact y is given *explicitly* at once. In the next place, the exponential function which is assumed for the solution of these equations, is derived from the binomial factors of operation, $\left(a_1-\dfrac{d}{dx}\right)$ &c., and as there are n factors

in an equation of the n^{th} order, there will be n exponential functions in the complete solution. Lastly, the equation

$$f\left(\frac{d}{dx}\right) y = X + 0$$

may be derived from the equation

$$f\left(\frac{d}{dx}\right) y = 0,$$

by differentiation only; for in operating with each factor of the form $\left(\frac{d}{dx} - a\right)^{-1}$ on X, we have only to expand according to powers of $\frac{d}{dx}$, and perform the operations indicated, and then add a term which must be the same as the term [30]
arising from the corresponding operation in the equation

$$f\left(\frac{d}{dx}\right) y = 0.$$

The application of this method to linear differential equations with variable coefficients is attended with considerable difficulty, and indeed neither Brisson nor Cauchy seem to have made any progress in the solution of these equations. There are, however, some which can be thus integrated, but we shall defer to a future number any observations we have to make on them, as well as the application of the same method to equations of finite and mixed differences, in which it is probably more useful than in differential equations.

But, before leaving the subject, we would say a few words on the legitimacy of the processes employed in this method. In the preceding pages we have spoken of treating the symbols of operation like those of quantity, so that at first sight it would appear as if the principles on which the method is founded, were drawn only from analogy. But a little consideration will show that this is not really the case, and that the reasoning on which we proceed is perfectly strict and logical. We have spoken as if there were a distinction between what are usually called symbols of operation, and those which are called symbols of quantity. But we might with perfect propriety call these last also symbols of operation. For instance, x is the operation designated by (x) performed on unity, x^n is the same operation performed n times in succession on unity, $a + x$ is the operation $(a + x)$ performed on unity, $(a + x)^n$ is the operation $(a + x)$ performed n times in succession on unity. By the phrase "in succession" is to be

understood, that the operations are performed, so to speak, successively one on the back of the other; and perhaps it would be better to say, that the operation (x) is repeated n times on unity. And in this x^n is to be distinguished from nx, which represents that n of the operations (x) on unity are taken simultaneously. In the same way as $a\,(1)$ represents the operation (a) performed on (1), $a\,(x)$ would represent the same operation performed on x, and $a^n\,(x)$ would represent the operation repeated n times on (x). These operations are usually written ax, a^nx.

If, then, we take this view of what are usually called symbols of quantity, we shall have little difficulty in seeing the correctness of the principle by which other operations, such as we represent by $\left(\dfrac{d}{dx}\right)$, (Δ), &c., are treated in the same way as a, b, &c. For whatever is proved of the latter symbols, from the known laws of their combination, must be equally true of all other symbols which are subject to the same laws of combination. Now the laws of the combinations of the symbols a, b, &c. are, that

$$a^m \,.\, a^n x = a^{m+n} \,.\, x \,\ldots\ldots\ldots\ldots\ldots (1),$$

$$a\,\{b\,(x)\} = b\,\{a\,(x)\} \,\ldots\ldots\ldots\ldots (2),$$

and $$a\,(x) + a\,(y) = a\,(x+y) \,\ldots\ldots\ldots\ldots (3).$$

And if f, f_1, &c. be any other general symbols of operation (f and f_1 being of the same kind) subject to the same laws of combination, so that

$$f^m . f^n\,(x) = f^{(m+n)}\,(x) \quad \ldots\ldots\ldots\ldots (1),$$

$$f\,\{f_1\,(x)\} = f_1\,\{f\,(x)\} \,\ldots\ldots\ldots\ldots (2),$$

[31] and $$f\,(x) + f\,(y) = f\,(x+y) \,\ldots\ldots\ldots\ldots (3).$$

Then, whatever we may have proved of a, b, &c. depending on these three laws, must necessarily be equally true of f, f_1, &c.

Now we know that the symbol d is subject to these laws for

$$d^m \,.\, d^n\,(x) = d^{m+n}\,(x),$$

$$\frac{d}{dx}\left\{\frac{d}{dy}\,(z)\right\} = \frac{d}{dy}\left\{\frac{d}{dx}\,(z)\right\} \ldots\ldots (2),$$

$$d\,(x) + d\,(y) = d\,(x+y),$$

and the same is true for the symbol Δ.

Hence the binomial theorem (to take a particular case) which has been proved for (a) and (b) is equally true for

$\left(\frac{d}{dx}\right)$ and $\left(\frac{d}{dy}\right)$: so that we require no farther proof for the proposition, that when u is a function of two independent variables x and y,

$$d^n(u) = \left(\frac{d}{dx}dx + \frac{d}{dy}dy\right)^n u = \frac{d^n u}{dx^n}dx^n + n\frac{d^{n-1}}{dx^{n-1}}\frac{d}{dy}udx^{n-1}dy + \ldots$$

But this reasoning will not apply in the case of those functions where the same laws do not hold. For instance, if we take the function log, we have not the condition

$$\log(x) + \log(y) = \log(x+y).$$

But $\qquad \log(x) + \log(y) = \log(xy).$

Consequently the binomial theorem will not hold for this function, though a binomial theorem might possibly be deduced for it, if the expressions did not become so complicated as to be unmanageable.

We have as yet only considered the combinations of operations of one kind, but in the preceding pages we frequently made use of operations of different kinds together, as in the expression $\left(\frac{d}{dx} - a\right)$. Now so long as each of the operations is subject to the same laws, and that they are independent, that is to say, that the one symbol is not supposed to act on the other, the same deductions will follow as when the operations are of the same kind. Hence we assumed that as the expression

$$x^n + Ax^{n-1} + Bx^{n-2} + \&c. + S$$

can be resolved into the factors

$$(x - a_1),\ (x - a_2),\ (x - a_3),\ \&c., \qquad [32]$$

the expression

$$\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \&c. + S$$

can be resolved into the factors

$$\left(\frac{d}{dx} - a_1\right),\ \left(\frac{d}{dx} - a_2\right), \ldots \left(\frac{d}{dx} - a_n\right),$$

which is the foundation of the method we have explained.

But if we have united together such symbols as $\left(\frac{d}{dx} + x\right)$, the same result will not hold. For though (x) is an operation of the same kind as (a), yet it bears a different relation to

$\left(\frac{d}{dx}\right)$, as by the nature of this last operation it affects the operation (x), so that

$$x\left\{\frac{d}{dx}(z)\right\} \text{ is not equal to } \frac{d}{dx}\{x(z)\},$$

or the second law of combination does not hold with regard to these symbols of operation, and, consequently, theorems for other symbols deduced from this law are not true for such symbols as $\left(\frac{d}{dx}\right)$ and (x) together. It is this peculiarity with regard to the combinations of the symbols (x) and $\frac{d}{dx}$ which gives rise to the difficulty in the solution of linear equations with variable coefficients.

Since this article was written, we have learnt that a report by *Cauchy* on *Brisson's* Memoirs, which appears to have been favourable, was rejected by the Academy of Sciences. We know not for what reason.

D. F. G.

SOLUTION OF TWO PROBLEMS IN ANALYTICAL GEOMETRY.*

In one of the Problem papers for 1836 there is given the following problem: To draw a tangent to a curve of the second order from a point P without it. From P draw any two lines, each cutting the curve in two points. Join the points of intersection two and two, and let the points in which the joining lines (produced if necessary) cross each other be joined by a line which will in general cut the curve in two points A, B. PA, PB, are tangents at A and B. [33] This problem admits of a very elegant solution, which is applicable to many similar questions, and which we shall therefore lay before our readers. Taking the two lines drawn from P as coordinate axes, the equation to the curve is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \text{ . . . . . . } (1).$$

Let the curve cut the axis of x in points M, M', and the axis of y in points N, N', and let $PM = a$, $PM' = a'$, $PN = b$, $PN' = b'$.

* These solutions were contributed by Mr. Archibald Smith. The method followed is due to the late Richard Stevenson, Fellow of Trinity College.

The equation to the line joining MN is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

the equation to the line joining $M'N'$ is

$$\frac{x}{a'} + \frac{y}{b'} = 1;$$

and, as at their intersection we may combine their equations in any manner, adding them we get

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \quad \ldots\ldots\ldots \quad (2).$$

Again, the equation to the line joining MN' is

$$\frac{x}{a} + \frac{y}{b'} = 1,$$

and the equation to the line joining $M'N$ is

$$\frac{x}{a'} + \frac{y}{b} = 1;$$

at their intersection

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \ldots\ldots\ldots\ldots (3),$$

which is identical with (2), and therefore is the equation to the line joining the points of intersection.

If, now, in equation (1) we make $y = 0$, we get

$$Ax^2 + Dx + F = 0,$$

as the equation for determining a and a'. And, by the theory of equations,

$$a + a' = -\frac{D}{A}, \quad aa' = \frac{F}{A};$$

therefore $$\frac{1}{a} + \frac{1}{a'} = -\frac{D}{F}.$$

Similarly, $$\frac{1}{b} + \frac{1}{b'} = -\frac{E}{F};$$

hence equations (2) and (3) become

$$Dx + Ey + 2F = 0 \quad \ldots\ldots\ldots\ldots \quad (4).$$

If now we were to change the coordinate axes, retaining [34] the same origin, we should have to make in (1) substitutions of the form

$$x = mx' + ny',$$
$$y = m'x' + n'y'.$$

From the form of these it appears, that those terms which are of the second degree in x and y, would not in their changes affect the other terms; for every term involving x^2, xy, and y^2, would after the change involve only x'^2, $x'y'$, and y'^2. Similarly, the terms which are of the first degree would also change independently. And it is clear that the constant term F would experience no change at all. Now, if we were to make the substitutions in equation (4) the term $2F$ would, as before, remain the same, and the terms $Dx + Ey$ would suffer the same change as the same terms in the equation to the curve. From this it appears, that if the lines PMM', PNN', change their position so that the coordinates are altered, the equation (4), when deduced from the transformed equation (1), will be the same as when the transformation is effected directly on itself; which shows that the line represented by (4) remains fixed in position while the coordinate axes are changed. Now the coordinate axes, cutting the curve each in two points, as they change their position, will ultimately become tangents, and this evidently at the points in which the line, which we have shown to be fixed in position, cuts the curve. Hence follows the method given in the problem.

In one of the problem papers for 1835, the following problem, which may be solved by the same principle, is given: If AB, $A'B'$, be any two chords in a surface of the second order, which intersect in a fixed point, the locus of the intersection of AA', BB', is a plane.

Take O the point of intersection of AB, $A'B'$ as origin, OAB as the axis of x, $OA'B'$ as the axis of y. Put $OA = a$, $OB = a'$, $OA' = b$, $OB' = b'$.

The equation to the surface is

$$Ax^2 + A'y^2 + A''z^2 + Byz + B'xz + B''xy$$
$$+ Cx + C'y + C''z + E = 0 \ldots\ldots\ldots (1);$$

the equation to the line AA' is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

the equation to the line BB' is

$$\frac{x}{a'} + \frac{y}{b'} = 1.$$

At their point of intersection we have, by adding the equations,

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \ldots\ldots\ldots (2).$$

If, now, in equation (1) we make $z=0$, $y=0$, we have [35]

$$Ax^2 + Cx + E = 0$$

as the equation for determining a and a'; whence

$$a + a' = -\frac{C}{A}, \quad aa' = \frac{E}{A};$$

therefore $$\frac{1}{a} + \frac{1}{a'} = -\frac{C}{E}.$$

Similarly, we should find

$$\frac{1}{b} + \frac{1}{b'} = -\frac{C'}{E};$$

whence equation (2) becomes

$$Cx + C'y + 2E = 0.$$

Now this equation may be considered as the equation of the line in which the plane, whose equation is

$$Cx + C'y + C''z + 2E = 0 \;\ldots\ldots\ldots\ldots (3),$$

cuts the plane of xy.

And, as in the last problem, we may show that the plane represented by equation (3) remains fixed in position; so that the locus of the intersection of AA', BB', is a plane fixed in space, and determined by the equation

$$Cx + C'y + C''z + 2E = 0.$$

R. S.

ON THE PRINCIPAL AXES OF ROTATION.

THE moment of inertia of a system about an axis, which passes through the origin of coordinates, and makes angles α, β, γ, with the axes of x, y, z, is equal to

$$f \sin^2 \alpha + g \sin^2 \beta + h \sin^2 \gamma$$
$$- 2F \cos \beta \,.\, \cos \gamma - 2G \cos \gamma \,.\, \cos \alpha - 2H \cos \alpha \,.\, \cos \beta,$$

where $f = \int x^2 dm$, $\quad g = \int y^2 dm$, $\quad h = \int z^2 dm$,

and $F = \int yz\, dm$, $\quad G = \int zx\, dm$, $\quad H = \int xy\, dm$.

This expression will be much simplified if we can assign to the coordinate axes such a position as shall render the corresponding values of F, G, and H equal to zero.

To discover whether this be possible, let us suppose the system referred to three rectangular axes of coordinates

x', y', z', which make angles with the axes of x, y, z, whose cosines are a, b, c, a', b', c', and a'', b'', c'' respectively. Be-
[36] tween these nine cosines there are six equations of condition, so that there remain but three to be satisfied in order to determine the position of the new axes. We require then that $\int y'z'dm = 0$, $\int z'x'dm = 0$, $\int x'y'dm = 0$.

Now we have
$$x = ax' + a'y' + a''z' \ldots\ldots\ldots\ldots (1)$$
$$y = bx' + b'y' + b''z' \ldots\ldots\ldots\ldots (2)$$
$$z = cx' + c'y' + c''z' \ldots\ldots\ldots\ldots (3)$$
and
$$x' = ax + by + cz \ldots\ldots\ldots\ldots (4).$$

Multiplying (1) by $x'dm$ and integrating, we have
$$\int xx'dm = a\int x'^2 dm,$$
the other terms vanishing by the conditions.

Also, multiplying (4) by xdm and integrating, we have
$$\begin{aligned}\int xx'dm &= a\int x^2 dm + b\int xy\,dm + c\int xz\,dm \\ &= af + bH + cG \\ &= a\int x'^2 dm\end{aligned}$$
by the last equation.

Putting $\int x'^2 dm = X$, and transposing the terms of the equation, we have
$$a(f - X) + bH + cG = 0 \ldots\ldots\ldots\ldots (5).$$

Treating the equations (2) and (4), (3) and (4), in the same manner, putting only y and z respectively in place of x, we obtain
$$aH + b(g - X) + cF = 0 \ldots\ldots\ldots (6)$$
$$aG + bF + c(h - X) = 0 \ldots\ldots\ldots (7),$$
and then eliminating the quantities a, b, and c from equations (5), (6), and (7), by the method of cross multiplication, we obtain the equation
$$(f - X)(g - X)(h - X)$$
$$- F^2 f - X) - G^2(g - X) - H^2(h - X) - 2FGH = 0,$$
which is that arrived at p. 267, Whewell's *Dynamics*. We should have obtained the same equation for $Y = \int y'^2 dm$, or for $Z = \int z'^2 dm$, and therefore, as it may be shown that the three roots of the cubic are real, they are the values of X, Y, and Z.

P.

ANALYTICAL GEOMETRY OF THREE DIMENSIONS. NO. I. [37]

THE series of articles which we intend to give on this subject are chiefly designed to exhibit the advantages of mathematical symmetry. The French writers have commonly been more attentive to this than our own, and it may be seen frequently exemplified in Leroy's *Analyse appliquée a la Géometrie des trois dimensions*. But neither he, nor any other person that we know of, has made any use of the following symmetrical form of the equations to the straight line, though he mentions it, p. 17, 2nd edit.

1. Let x', y', z', be the coordinates of any fixed point through which the line passes; and λ, μ, ν, the angles which it makes with the axes of coordinates, supposed rectangular; then

$$\frac{x-x'}{\cos\lambda}=\frac{y-y'}{\cos\mu}=\frac{z-z'}{\cos\nu}\ \ldots\ldots\ldots\ (1)$$

are the equations to the line.

If the angles between the coordinates be any whatever, and if l, m, n, be the ratios of the projections of any portion of the line on the axes of coordinates, to that portion, the projections being made by planes parallel to the coordinate planes, then the equations to the line are

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}.$$

The demonstration of this is very simple.

Let r be the length of the portion of the line between the points (x', y', z'), (x, y, z), then the projections of r are

$$lr,\ mr,\ nr.$$

But these projections are also

$$x-x',\quad y-y',\quad z-z'.$$

Equating these values, we obtain

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r\ \ldots\ldots\ (2).$$

It will frequently be convenient to introduce the quantity r in investigations.

2. If the angles between yz, zx, xy, be denoted by yz, zx, xy, respectively,*

$$r^2=x^2+y^2+z^2+2yz\cos yz+2zx\cos zx+2xy\cos xy\ldots(3),$$

* See p. 94.

therefore

$$l^2+m^2+n^2+2mn\cos yz+2nl\cos zx+2lm\cos xy=1\ldots(4)$$

is the relation connecting l, m, n.

[38] 3. It is evident that if L, M, N be any quantities respectively proportional to l, m, n, the equations to the line may be written

$$\frac{x-x'}{L}=\frac{y-y'}{M}=\frac{z-z'}{N}.$$

4. We shall not stop to prove the expression for the cosine of the angle contained between two lines, in terms of the angles which they make with a system of rectangular coordinates, but proceed to the more general proposition of finding the angle between two given lines, the angles between the coordinates being any whatever.

Let the equations to two straight lines parallel to the given lines and passing through the origin, be

$$\left.\begin{aligned}\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r\\ \frac{x'}{l'}=\frac{y'}{m'}=\frac{z'}{n'}=r'\end{aligned}\right\}\ldots\ldots\ldots\ldots(5),$$

and let θ be the angle between them.

Then the square of the distance between the extremities of the lines r, r', is $r^2-2rr'\cos\theta+r'^2\ldots\ldots\ldots\ldots(6)$;

but it is also the same as the square of the distance between the points (x, y, x), (x', y', z'), which by the expression (3), § 2, is

$$\begin{aligned}&(x-x')^2+(y-y')^2+(z-z')^2\\ &+2(y-y')(z-z')\cos yz+2(z-z')(x-x')\cos zx\\ &+2(x-x')(y-y')\cos xy\\ &=r^2+r'^2-2\{xx'+yy'+zz'+(yz'+y'z)\cos yz\\ &+(zx'+z'x)\cos zx+(xy'+x'y)\cos xy\}\ldots(7).\end{aligned}$$

Equating the expressions (6) and (7), substituting for x, y, z, x', y', z', in terms of r and r' from equations (5), and reducing, we get

$$\begin{aligned}\cos\theta=ll'+mm'+nn'+(mn'+m'n)\cos yz+(nl'+n'l)\cos zx\\ +(lm'+l'm)\cos xy\ldots\ldots\ldots\ldots(8),\end{aligned}$$

which is one expression for $\cos\theta$.

If the two lines coincide, so that $\theta=0$, $l'=l$, $m'=m$, and $n'=n$, equation (8) becomes the same as the equation (4) expressing the relation between l, m, n.

If the axes be rectangular, cos yz, cos zx, cos xy, are each $= 0$, and (8) becomes

$$\cos\theta = ll' + mm' + nn',$$

l, m, n, l', m', n' becoming in this case the cosines of the angles which the two lines make with the axes.

5. Let the angles which the first line makes with the axes, in the general case, be λ, μ, ν, we may deduce from (8) the values of λ, μ, ν, in terms of l, m, n.

For suppose the second line to coincide with the axis [39] of x, then θ becomes λ; and by considering the signification of these quantities, it will be seen that l' becomes 1, m' and n' each become 0. Hence (8) reduces to

$$\left.\begin{aligned} \cos\lambda &= l + n\cos zx + m\cos xy. \\ \text{In like manner,}\quad \cos\mu &= m + n\cos yz + l\cos xy, \\ \text{and}\quad \cos\nu &= n + m\cos yz + l\cos zx. \end{aligned}\right\}\ldots\ldots(9).$$

6. We may now obtain a shorter expression for $\cos\theta$. Multiply equations (9) by l', m', n' respectively, and add the products; then the second member of the resulting equation will be identical with the second member of (8), therefore

$$\cos\theta = l'\cos\lambda + m'\cos\mu + n'\cos\nu \ldots.\ (10).$$

This equation is remarkable for being of the same form as when the axes are rectangular. By supposing the two lines to coincide, we obtain from it

$$l\cos\lambda + m\cos\mu + n\cos\nu = 1 \ldots\ldots\ (11),$$

a relation between these quantities independent of the angles between the axes.

7. The values of l, m, n, in terms of λ, μ, ν, may be easily found from equations (9), by cross multiplication.

We may also substitute the values of $l'\ m', n'$, in terms of λ', μ', ν', in (10), and thus obtain an expression for $\cos\theta$ in terms of the angles which the lines make with the axes.

8. From (10) we obtain for the condition, that a line which makes angles λ, μ, ν with the axes, may be at right angles to a line whose equations are

$$\frac{x - x'}{l'} = \frac{y - y'}{m'} = \frac{z - z'}{n'},$$

$$l'\cos\lambda + m'\cos\mu + n'\cos\nu = 0 \ldots\ldots(12).$$

Another form of the condition may be obtained from (8).

9. To obtain the equation to the plane, we shall consider it as generated by the motion of a straight line which always meets a fixed straight line, and remains parallel to a given position.

Let the equations to the generating line be

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} = r,$$

and those to the fixed line,

$$\frac{x' - a}{l'} = \frac{y' - b}{m'} = \frac{z' - c}{n'} = r'.$$

Hence,

$$x - x' = lr, \quad y - y' = mr, \quad z - z' = nr;$$

$$x' - a = l'r', \quad y' - b = m'r', \quad z' - c = n'r'.$$

[40] Adding these equations, so as to eliminate x', y', z',

$$\left.\begin{aligned} x - a &= lr + l'r' \\ y - b &= mr + m'r' \\ z - c &= nr + n'r' \end{aligned}\right\} \ldots\ldots\ldots\ldots (13).$$

Let α, β, γ be the angles which a line perpendicular to both the former, makes with the axes, so that, according to (12),

$$\left.\begin{aligned} l \cos \alpha + m \cos \beta + n \cos \gamma = 0, \\ l' \cos \alpha + m' \cos \beta + n' \cos \gamma = 0, \end{aligned}\right\} \ldots\ldots (14);$$

then, multiplying equations (13) by $\cos \alpha$, $\cos \beta$, $\cos \gamma$, respectively, and adding, r and r' disappear, and

$$(x - a) \cos \alpha + (y - b) \cos \beta + (z - c) \cos \gamma = 0 \,..(15),$$

which is the equation to the plane. This equation proves Euclid, book XI. prop. 4, for if ρ be the distance of the points (x, y, z) and (a, b, c), $x - a$, $y - b$, $z - c$, are the projections of ρ on the axes, and the equation

$$\frac{x - a}{\rho} \cos \alpha + \frac{y - b}{\rho} \cos \beta + \frac{z - c}{\rho} \cos \gamma = 0$$

shews that the line which makes angles α, β, γ with the axes is perpendicular to ρ, that is, to any line which meets it in that plane, and therefore it is perpendicular to the plane.

10. It is easy to derive from (15) other useful forms of the equation to the plane. We may suppose a, b, c, to be the coordinates of the point where the perpendicular from the origin on the plane meets it; then, if the length of the per-

pendicular be δ, we have by (11), since a, b, c are the projections of δ upon the axes,

$$\frac{a}{\delta}\cos\alpha + \frac{b}{\delta}\cos\beta + \frac{c}{\delta}\cos\gamma = 1,$$

whence $$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta\ \ldots\ldots (16).$$

11. The sine of the angle between the line whose equations are

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n},$$

and the plane whose equation is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta,$$

is the same as the cosine of the angle between the line and the perpendicular to the plane, which, by (10), is

$$l\cos\alpha + m\cos\beta + n\cos\gamma.$$

If the line be parallel to the plane, this must be equal to 0. If the line be situated in the plane, we must have, in addition,

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma = \delta.$$

When the coordinates are rectangular, the conditions that the line may be perpendicular to the plane, are

$$l = \cos\alpha,\quad m = \cos\beta,\quad n = \cos\gamma.$$

12. To find the perpendicular distance from a given point (x', y', z') to the plane represented by [41]

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta.$$

Let a parallel plane be drawn through the point (x', y', z'), then its equation will be

$$(x-x')\cos\alpha + (y-y')\cos\beta + (z-z')\cos\gamma = 0.$$

The required perpendicular will evidently be equal to the difference between the perpendiculars from the origin on these two planes

$$= (x'\cos\alpha + y'\cos\beta + z'\cos\gamma) \sim \delta.$$

13. To find the shortest distance between two lines, whose equations are

$$\left.\begin{aligned} \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r \\ \frac{x'-a'}{l'} = \frac{y'-b'}{m'} = \frac{z'-c'}{n'} = r' \end{aligned}\right\} \ldots\ldots (17),$$

referred to rectangular coordinates.

If D be the distance of two points in the two lines,

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

Now x, y, z are each functions of r, and x', y', z' of r'; but r and r' are independent, therefore, when D is a minimum,

$$(x - x')\,dx + (y - y')\,dy + (z - z')\,dz = 0,$$

$$(x - x')\,dx' + (y - y')\,dy' + (z - z')\,dz' = 0.$$

But
$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n} = dr,$$

$$\frac{dx'}{l'} = \frac{dy'}{m'} = \frac{dz'}{n'} = dr'.$$

Substituting in the preceding equations,

$$\left.\begin{array}{l} l(x - x') + m(y - y') + n(z - z') = 0 \\ l'(x - x') + m'(y - y') + n'(z - z') = 0 \end{array}\right\}\ldots(18).$$

From these equations we see that the coordinates of the extremities of the shortest distance satisfy the equations to two planes, respectively perpendicular to the two given lines. The line of the shortest distance is therefore the intersection of these two planes, and therefore perpendicular to both the given lines.

The six equations (17) and (18) determine the values of the six coordinates x, y, z, x', y', z'. The best way to solve them would be to substitute in (18) the values of the coordinates in terms of r and r' obtained from (17): thus we should have two simple equations for determining r and r', the values of which being found, those of the coordinates would be known.

[42] The value of the least distance may easily be determined by supposing two planes, parallel to one another, to pass through the given lines. These planes will therefore be perpendicular to the least distance, and their equation will be

$$(x - a)\cos\alpha + (y - b)\cos\beta + (z - c)\cos\gamma = 0,$$

$$(x - a')\cos\alpha + (y - b')\cos\beta + (z - c')\cos\gamma = 0,$$

$\cos\alpha$, $\cos\beta$, and $\cos\gamma$ being determined by the equations

$$l\cos\alpha + m\cos\beta + n\cos\gamma = 0,$$

$$l'\cos\alpha + m'\cos\beta + n'\cos\gamma = 0,$$

$$(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

The least distance of the lines will be the perpendicular distance of these parallel planes, or the difference of the perpendiculars upon them from the origin, which is

$$\pm\left\{(a - a')\cos\alpha + (b - b')\cos\beta + (c - c')\cos\gamma\right\}.$$

S. S. G.

NOTE ON THE THEORY OF THE SPINNING-TOP.

THE manner in which friction causes a spinning-top to raise itself into a vertical position, has never, as far as I know, been distinctly shown. Euler gives the following explanation, which will be found in a note to Whewell's *Dynamics*, p. 324. "The friction will perpetually retard the motion of the apex of the instrument, and at last reduce it to rest. If this happen before the top fall, it must then be spinning in such a position that the point can remain stationary; but this cannot be if it be inclined. Hence it must have a tendency to erect itself into a vertical position." This reasoning is not only of the most vague and inconclusive kind, but is remarkable as being directly the reverse of the truth. For when the friction acts so as to retard the apex, it tends to make the top fall; and it is only when the friction accelerates the apex, that it causes the top to raise itself into a vertical position.

It is well known, that if a body have velocities communicated to it about different axes, which are represented by distances taken along these axes, proportional to the velocities, all the rotations being considered as "right-handed," and "left-handed" rotations being represented by taking the distances in the negative direction, then the axis and magnitude of the resultant rotation will be represented by the resultant of these lines combined as lines representing forces.

Suppose, now, a top, whose apex is a mathematical [43]
point, to be spinning on a smooth horizontal plane in an inclined position, and, for distinctness of conception, let this be the plane of xy, and let the axis be in the plane yz.

The action of gravity will tend to make it fall, by giving it a motion of rotation about an axis parallel to that of x. The angular velocity this would communicate in an instant of time may be represented by a line in the direction of $-x$. The combination of these two will cause the instantaneous axis to move a little towards the axis of $-x$, and the axis of figure to follow *nearly* in the same direction, and the effect will be a precessional motion of the top in the same direction as the rotation, combined with a nutation which we do not here consider. Consequently, the centre of gravity remaining in the same vertical line, the apex will describe a circular path "with the Sun."

If the plane be rough, the apex will endeavour to describe the same path, but will be constantly retarded by friction,

which at the instant we consider will be a force pulling the apex in the direction of $-x$, which will tend to produce a rotation about an axis perpendicular to a plane passing through the direction and the centre of gravity, or lying between the axes of y and $-z$. The combination of this with the original rotation will thus cause the axis of the top to approach the axis of y, or to fall.

Hitherto we have supposed the apex of the top to be a mathematical point, which can never be exactly the case; the real apex will be a surface. If we consider the top as terminated by a portion of a sphere, it will be seen that the top spins on no particular point, but that each point in the circumference of a small circular curve is successively in contact with the plane, the diameter of the curve increasing with the inclination of the top.

Conceive, now, the top spinning in its original position, and with no precessional motion, the apex will endeavour to roll along the axis of x; if it is prevented from rolling, there will be a rubbing friction tending to pull the apex in that direction.

Suppose, now, the top to spin freely—the precessional motion makes the apex move towards x: if this velocity is equal to the rolling velocity, there will be no friction called into action, and the top will spin as if on a smooth plane. If the precessional motion be greater than the rolling motion, there will be a retarding friction which will cause the top to fall. *But if the rolling motion is quicker than the precessional, there will be an accelerating friction which will tend to raise the top.*

It is easy to show, by means of a teetotum, that this theory agrees with experiment. If the apex be cut to a point, no velocity we can communicate will make it spin upright; if the apex is rounded, as it usually is, the instrument will rise at first, and then fall gradually, the increased diameter of the circle in which it touches the plane compensating for the diminishing velocity of rotation and the increasing velocity of
[44] precession: or if the end be a cylinder cut perpendicularly to its axis, in which case this compensation does not take place, it will be found that the teetotum spins upright for some time, and then falls very suddenly as soon as the direction of friction changes.

A. S.

ON THE SOLUTION OF CERTAIN TRIGONOMETRICAL EQUATIONS.

There are several trigonometrical equations whose roots can be readily obtained by taking into consideration their connexion with the general binomial equation whose last term is unity. If, for example, we have the equation

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos(n-1)\theta = 0,$$

we can find its roots by means of the equation

$$x^{2n} - 1 = 0.$$

We know that the roots of this last equation are of the form

$$\pm 1, \cos\phi \pm \sqrt{(-1)}\sin\phi, \cos 2\phi \pm \sqrt{(-1)}\sin 2\phi \ldots$$
$$\cos(n-1)\phi \pm \sqrt{(-1)}\sin(n-1)\phi.$$

Now as the equation wants the second term, the sum of its roots must be 0; and as the possible and impossible parts do not affect each other, they must be separately equal to 0: also, as the roots $+1$ and -1 destroy each other, there remains

$$\cos\phi + \cos 2\phi + \cos 3\phi + \ldots \cos(n-1)\phi = 0.$$

Comparing this with the original equation, we see that the former will be satisfied by making $\theta = \phi$. But to determine ϕ, we have the equation

$$\{\cos\phi + \sqrt{(-1)}\sin\phi\}^{2n} - 1 = 0,$$

or

$$\cos 2n\phi + \sqrt{(-1)}\sin 2n\phi = 1.$$

Whence, as the possible and impossible parts are independent, $\cos 2n\phi = 1$, $\sin 2n\phi = 0$;

which give

$$2n\phi = 2m\pi,$$

or

$$\phi = \frac{m\pi}{n},$$

where m has any integer value from 0 to n, making in all $n+1$ values of ϕ. But two of these cannot be taken as values of θ, as we excluded the roots $+1$ and -1, which correspond to the values 0 and n of m. So that θ will be found from the equation

$$\theta = \frac{m\pi}{n},$$

where m has any value from 1 to $n-1$, making in all $n-1$ values of θ which answer the given equation. In exactly [45]
the same way we might shew how to solve the equation

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta = 0.$$

For its roots would be deduced from the equation

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{2n} + 1 = 0,$$

or

$$\cos 2n\theta + \sqrt{(-1)}\sin 2n\theta = -1;$$

which gives

$$\cos 2n\theta = -1, \quad \sin 2n\theta = 0.$$

Therefore

$$2n\theta = (2m+1)\pi,$$

and

$$\theta = \frac{(2m+1)}{2n}\pi;$$

where m has any value from 0 to $n-1$, giving on the whole n values of θ which satisfy the equation.

Similarly, the equations

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = 0,$$

and

$$\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\theta = 1,$$

may be solved by means of the equations

$$x^{2n+1} - 1 = 0,$$

and

$$x^{2n+1} + 1 = 0.$$

For the first we shall have

$$\theta = \frac{2m\pi}{2n+1},$$

where m has n values from 1 to n.

For the second we shall have

$$\theta = \frac{2m+1}{2n+1}\pi,$$

where m has n values from 0 to $n-1$.

The same method may be extended to other equations. For the roots of $x^{2n} - 1 = 0$ being of the form

$$\cos\phi + \sqrt{(-1)}\sin\phi.$$

If we multiply each by $\cos a + \sqrt{(-1)}\sin a$, it becomes

$$\cos(a+\phi) + \sqrt{(-1)}\sin(a+\phi).$$

But the sum of the roots will still remain equal to 0 when multiplied by $\cos a + \sqrt{(-1)}\sin a$; and taking away the terms which destroy each other, there will remain

$$\cos(a+\phi) + \cos(a+2\phi) + \cos(a+3\phi) + \ldots + \cos\{a+(n-1)\phi\} = 0;$$

consequently the equation

$$\cos(a+\theta) + \cos(a+2\theta) + \ldots + \cos\{a+(n-1)\theta\} = 0,$$

will be satisfied by the same values of θ as the first of the given equations: and similarly we might proceed with the others.

D. F. G.

MATHEMATICAL NOTES. [46]

UNDER this head we propose to insert new demonstrations of known theorems, solutions of interesting problems, and in general short notices of methods of mathematical investigation which are likely to be useful to all classes of students in this university.

1. *Elimination by means of Cross Multiplication.*—This is a convenient mnemonic rule for elimination, applications of which continually occur in various branches of analysis; and references to which have been frequently made in different articles of this number.

If we have three symmetrical equations

$$Ax + By + Cz = D \dots\dots\dots\dots (1),$$
$$A_1x + B_1y + C_1z = D_1 \dots\dots\dots\dots (2),$$
$$A_2x + B_2y + C_2z = D_2 \dots\dots\dots\dots (3),$$

between which we wish to eliminate y and z, we can perform the operation at once by the following rule:

$$\begin{aligned}\text{Multiply } (1) \text{ by } & B_1C_2 - B_2C_1,\\ (2) \text{ by } & B_2C - BC_2,\\ (3) \text{ by } & BC_1 - B_1C,\end{aligned}$$

and add. It will be found on trial, that the terms involving y and z disappear, and we obtain

$$\{A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C)\}x$$
$$= D(B_1C_2 - B_2C_1) + D_1(B_2C - BC_2) + D_2(BC_1 - B_1C),$$

from which x is known. And in a similar manner we might determine y and z by eliminating x and z, and x and y successively.

If $D = 0$, $D_1 = 0$, $D_2 = 0$, the second side of the equation disappears, and as x divides out, we have, as the result of the elimination of x, y, z, the equation

$$A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C) = 0.$$

As examples of cases in which this method is of great advantage, we may notice the investigation of the cubic equation of condition for the existence of three principal axes of rotation, and of three principal diametral planes in surfaces of the second order, and the demonstration of the properties of conjugate diameters in these surfaces. It is also of great use, in finding the equation to the osculating plane, and the radius of absolute curvature, if all the expressions be put in a symmetrical form. The form of the multi-

pliers may be easily deduced by Lagrange's method of indeterminate multipliers, and their symmetry greatly facilitates the practical application of the method.

[47] σ.

2. *To find* $\dfrac{d\theta}{de}$ *and* $\dfrac{dr}{de}$ *in the Planetary Theory.*—The following method, by the introduction of a subsidiary quantity, simplifies greatly the analytical operations, and more particularly avoids a very troublesome integration. (*Airy*, p. 94; *Pratt*, p. 331.)

Taking the equations of elliptic motion,

$$nt = u - e \sin u \ldots\ldots\ldots\ldots\ldots (1),$$

$$\tan \frac{u}{2} = \frac{\sqrt{(1-e)}}{\sqrt{(1+e)}} \tan\left(\frac{\theta - \omega}{2}\right) \ldots\ldots\ldots (2).$$

Differentiate (1) with regard to e, considering t as constant, and take the logarithmic differential of (2) with regard to the same variable. Then we have

$$0 = \frac{du}{de}(1 - e\cos u) - \sin u \ldots\ldots\ldots\ldots (3),$$

$$\frac{1}{\sin u}\frac{du}{de} = -\frac{1}{1-e^2} + \frac{1}{\sin(\theta - \omega)}\frac{d\theta}{de} \ldots\ldots (4).$$

Eliminating $\dfrac{1}{\sin u}\dfrac{du}{de}$, and observing that

$$1 - e\cos u = \frac{1-e^2}{1 + e\cos(\theta - \omega)},$$

we get $$\frac{1}{\sin(\theta - \omega)}\frac{d\theta}{de} = \frac{2 + e\cos(\theta - \omega)}{1 - e^2};$$

whence $$\frac{d\theta}{de} = \frac{\sin(\theta - \omega)\{2 + e\cos(\theta - \omega)\}}{1 - e^2}.$$

Again, we have $r = a(1 - e\cos u)$.

Therefore $$\frac{dr}{de} = -a\cos u + ae\sin u\frac{du}{de}.$$

Eliminating $\dfrac{du}{de}$ by means of (3),

$$\frac{dr}{de} = -a\cos u + \frac{ae\sin^2 u}{1 - e\cos u} = -\frac{a\cos u + ae}{1 - e\cos u}.$$

Therefore $$\frac{dr}{de} = -\frac{a(\cos u - e)}{1 - e\cos u} = -a\cos(\theta - \omega).$$

π.

3. *Evolute to the Ellipse.*—The equation to the evolute of the ellipse may be found very readily by considering it as the locus of the ultimate intersection of consecutive normals.

Let
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

be the equation to the ellipse. Then the equation to a [48]
normal passing through a point x, y, will be

$$\frac{b^2(\beta - y)}{y} - \frac{a^2(\alpha - x)}{x} = 0,$$

or
$$\frac{b^2\beta}{y} - \frac{a^2\alpha}{x} = -(a^2 - b^2) \quad \ldots\ldots\ldots\ldots \quad (2),$$

where α and β are the coordinates of the normal itself. To find the locus of the ultimate intersection of the normals, we must differentiate considering α and β as constant, x and y as variable. We then have from equations (1) and (2)

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\frac{b^2\beta dy}{y^2} - \frac{a^2\alpha dx}{x^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (4):$$

$\lambda\,(3) + (4)$ gives, on equating to 0 the coefficients of each differential,

$$\frac{\lambda x}{a^2} = \frac{a^2\alpha}{x^2}, \quad \frac{\lambda y}{b^2} = -\frac{b^2\beta}{y^2}.$$

Multiply the first of these by x, and the second by y, and add. Then

$$\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \lambda = \frac{a^2\alpha}{x} - \frac{b^2\beta}{y} = a^2 - b^2.$$

Substituting

$$\frac{a^2 - b^2}{a^2}\,x = \frac{a^2\alpha}{x^2}, \quad \frac{a^2 - b^2}{b^2}\,y = -\frac{ba^2\beta}{y^2}.$$

Therefore
$$\frac{x^3}{a^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{y^3}{b^3} = -\frac{b\beta}{a^2 - b^2};$$

and these values of $\frac{x}{a}$ and $\frac{y}{b}$ being substituted in the equation to the ellipse, give

$$a^{\frac{2}{3}}\alpha^{\frac{2}{3}} + b^{\frac{2}{3}}\beta^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

γ.

[49] ELIMINATION BY INDETERMINATE MULTIPLIERS.

As we have made frequent use of the method of elimination by means of indeterminate multipliers, we shall give the theory of the process.

The solution of an extensive class of problems may be reduced to the determination of the maximum or minimum of a function U of n variables, $x_1, x_2, \ldots x_n$; these being in general not independent, but subject to certain equations of condition,
$$L_1 = 0, \quad L_2 = 0, \ldots L_r = 0.$$

The method simplest in theory, is to determine, by means of the r equations of condition, r of the quantities x, as for instance, $x_1, x_2, \ldots x_r$, in terms of the remaining x's: substituting these values, U will now be a function of the $n-r$ independent variables $x_{n-r} \ldots x_n$; and making the coefficient of each differential in dU equal to zero, we get $n-r$ equations, which, with the r equations of condition, determine $x_1, x_2, \ldots x_n$.

But in practice this method is rarely possible, and still more rarely convenient. Instead of this, we may consider each of the n quantities x to vary in U, their variations being no longer independent, but subject to the condition that in their varied state they must satisfy the r equations $L = 0$, or that $L_1 + dL_1 = 0$, &c.; and as $L_1, L_2, \ldots$ are $= 0$, the variations must be such as to satisfy the r equations
$$dL_1 = 0, \quad dL_2 = 0, \ldots \ldots dL_r = 0.$$

From these equations we may determine r of the variations in terms of the $(n-r)$ remaining; and substituting these in dU, we shall have, as before, the coefficients of the $(n-r)$ variations to equate to zero.

[50] The simplest method of effecting this is the following. The quantities $dL_1, dL_2, \ldots\ldots$ as well as dU, are linear in $dx_1, dx_2, \ldots\ldots$ being of the form
$$\frac{dL}{dx_1}dx_1 + \frac{dL}{dx_2}dx_2 \ldots \frac{dL}{dx_n}dx_n.$$
Multiplying them by $\lambda_1, \lambda_2, \ldots \lambda_r$ respectively, and adding them to dU, we have
$$dU + \lambda_1 dL_1 + \lambda_2 dL_2 \ldots\ldots + \lambda_r dL_r = 0,$$
an equation which will be of the form
$$M_1 dx_1 + M_2 dx_2 \ldots\ldots + M_n dx_n = 0,$$
in which each quantity $M = \dfrac{dU}{dx} + \lambda_1 \dfrac{dL_1}{dx} + \lambda_2 \dfrac{dL_2}{dx}$.

If we determine the r quantities λ by the conditions that they must make the coefficients of the variations $dx_1, dx_2, \ldots dx_r$ equal to zero, that is to say, if we determine them from the equations $M_1 = 0 \ldots M_r = 0$, we shall have effected our purpose. For by this means the variations $dx_1 \ldots dx_r$ are eliminated, and there remains

$$M_{r+1}\, dx_{r+1} \ldots\ldots + M_n\, dx_n = 0\,;$$

and as these $n - r$ variations are independent, their coefficients must be equal to zero.

Hence, if we determine $\lambda_1, \lambda_2, \ldots \lambda_r$ from the equations

$$M_1 = 0 \ldots\ldots M_r = 0,$$

we shall also have $M_{r+1} = 0 \ldots\ldots M_n = 0$.

Thus the r quantities λ and the n quantities x satisfy the $n + r$ equations

$$M_1 = 0,\quad M_2 = 0, \ldots M_n = 0,\quad L_1 = 0, \ldots L_r = 0.$$

And as it is indifferent which of the variations we suppose to have been eliminated, and which set of the equations $M = 0$ to have been used for the determination of the λ's, the most general way of stating the result is, that we have the $n + r$ equations

$$L_1 = 0,\quad L_2 = 0, \ldots L_r = 0,$$

$$\left.\begin{array}{l} \dfrac{dU}{dx_1} + \lambda_1 \dfrac{dL_1}{dx_1} + \lambda_2 \dfrac{dL_2}{dx_1} \ldots \lambda_r \dfrac{dL_r}{dx_1} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{dU}{dx_n} + \lambda_1 \dfrac{dL_1}{dx_n} + \lambda_2 \dfrac{dL_2}{dx_n} \ldots \lambda_r \dfrac{dL_r}{dx_n} = 0, \end{array}\right\} \ldots\ldots (1),$$

to determine the $n + r$ quantities $x_1, x_2 \ldots x_n, \lambda_1 \ldots \lambda_r$.

If U be homogeneous in $x_1, x_2 \ldots$, and if $L_1, L_2 \ldots$ consist of terms of not more than two different dimensions, there exists a simple relation between the quantities λ, which is very useful in many problems.

Suppose U to be homogeneous of u dimensions, $L_1, L_2,$ [51]
&c. to consist of terms of two different dimensions, so that the equations $L_1 = 0$, $L_2 = 0$, &c., may be put under the form

$$M_a + A = 0,\quad N_b + B = 0, \text{ \&c.},$$

where M_a is homogeneous of a dimensions, and A a constant.

Then, multiplying the equations (1) by $x_1, x_2, \ldots\ldots$ and adding, we have, by the property of homogeneous functions,

$$0 = uU + a\lambda_1 M_a + b\lambda_2 N_b + \text{\&c.}$$

or

$$uU = aA\lambda_1 + bB\lambda_2 + cC\lambda_3 + \text{\&c.}$$

We shall now proceed to give a few examples of the application of this method.

If we take the tangent plane at any point in a surface for the plane of xy, and the normal for the axis of z; and if r, s, t, according to the usual notation, represent the values of $\frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}$, at the point in question, α, β the cosines of the angles that a plane passing through the axis of z makes with the planes of xz, yz, and if ρ be the radius of curvature of the section of the surface made by this plane, it is known that

$$\frac{1}{\rho} = r\alpha^2 + 2s\alpha\beta + t\beta^2 \ldots\ldots\ldots\ldots (1),$$

with the relation $\alpha^2 + \beta^2 = 1 \ldots\ldots\ldots\ldots (2)$.

If we wish to find the direction of the section of greatest or least curvature, we may substitute $\sqrt{(1-\alpha^2)}$ for β; we have then

$$\frac{1}{\rho} = (r-t)\,\alpha^2 + 2s\alpha\,\sqrt{(1-\alpha^2)} + t,$$

from which we may determine the maximum and minimum values of ρ with the corresponding directions.

But without eliminating β, we may consider it as a function of α given by equation (2), and make α and β vary in (1), subjecting them to the condition of satisfying (2) in their varied state.

Hence
$$0 = (r\alpha + s\beta)\,d\alpha + (s\alpha + t\beta)\,d\beta \ldots\ldots (3),$$
$$0 = \alpha\,d\alpha + \beta\,d\beta \ldots\ldots (4).$$

If we eliminate $d\beta$, the coefficient of $d\alpha$ equated to zero will give a relation between α and β: to do this, multiply (4) by λ, and add to (3)

$$(r\alpha + s\beta + \lambda\alpha)\,d\alpha + (s\alpha + t\beta + \lambda\beta)\,d\beta = 0.$$

If, then, we determine λ from the condition

$$s\alpha + t\beta + \lambda\beta = 0 \ldots\ldots\ldots\ldots (5),$$

we must also have

$$r\alpha + s\beta + \lambda\alpha = 0 \ldots\ldots\ldots\ldots (6).$$

[52] These equations give us at once

$$s\frac{\alpha}{\beta} + t = -\lambda = r + s\frac{\beta}{\alpha},$$

or
$$\frac{\alpha^2}{\beta^2} - \frac{r-t}{s}\frac{\alpha}{\beta} - 1 = 0 \ldots\ldots\ldots\ldots (7),$$

which shows us, that the two directions of maximum and minimum curvature are at right angles to each other.

Multiplying by α and β, and adding,

$$r\alpha^2 + 2s\alpha\beta + t\beta^2 + \lambda(\alpha^2 + \beta^2) = 0,$$

or

$$\frac{1}{\rho} + \lambda = 0 \dots\dots\dots\dots\dots\dots (8).$$

Substituting this value, (5) and (6) may be put in the form

$$\left(\frac{1}{\rho} - t\right)\beta = s\alpha,$$

$$\left(\frac{1}{\rho} - r\right)\alpha = s\beta,$$

and multiplying together,

$$\left(\frac{1}{\rho} - r\right)\left(\frac{1}{\rho} - t\right) - s^2 = 0.$$

The relation $\frac{1}{\rho} + \lambda = 0$, corresponds to the relation

$$uU = \alpha a\lambda_1 + \text{\&c.},$$

which we have found in the general case.

It may be observed in this case, that if we merely wish to determine the relation between α and β, we may eliminate $d\beta$ at once and obtain the relation; and if we have three equations and three arbitrary variations, we should get the relation by eliminating two of them by the method of cross multiplication. An example of this will be found in the investigation in page 7 of our last Number.

For a second example, let us investigate the properties of conjugate diameters in surfaces of the second order.

Let

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1 \dots\dots\dots\dots\dots\dots (1)$$

be the equation of the surface referred to any three conjugate diameters, whose lengths are a', b', c'.

If l, m, n be the cosines of the angles which the axes make with each other, r any radius,

$$r^2 = x^2 + y^2 + z^2 + 2lyz + 2mxz + 2nxy \dots\dots (2).$$

When r coincides with one of the principal axes, it is a maximum or minimum, and we have the condition $dr = 0$.

If we determine z from (1), and substitute in (2), r [53] is a function of x and y only, and hence we should have

$$\frac{d(r)}{dx} = 0, \quad \frac{d(r)}{dy} = 0;$$

which, with (1) and (2), will serve to determine x, y, z, r.

This would be very difficult; but we may, as before, differentiate (1) and (2), considering x, y, z to vary.

We thus get

$$0 = (x + mz + ny)\, dx + (y + lz + nx)\, dy + (z + ly + mx)\, dz,$$

$$0 = \frac{x}{a'^2}\, dx + \frac{y}{b'^2}\, dy + \frac{z}{c'^2}\, dz.$$

If we multiply the second by λ and add to the first, and determine λ from the condition

$$\lambda \frac{z}{c'^2} + z + ly + mx = 0,$$

we have an equation in which dz has been eliminated: and as dx, dy are independent, their coefficients must also be equal to zero, or

$$\lambda \frac{y}{b'^2} + y + lz + nx = 0,$$

$$\lambda \frac{x}{a'^2} + x + mz + ny = 0.$$

If we multiply these by x, y, z respectively and add, we obtain, in consequence of (1) and (2),

$$\lambda + r^2 = 0.$$

Substituting this value of λ in the equations, they become

$$\left(1 - \frac{r^2}{a'^2}\right) x + ny + mz = 0,$$

$$nx + \left(1 - \frac{r^2}{b'^2}\right) y + lz = 0,$$

$$mx + ly + \left(1 - \frac{r^2}{c'^2}\right) z = 0.$$

Eliminating x, y, z between these by the method of cross multiplication, the resulting equation is

$$\left(1 - \frac{r^2}{a'^2}\right)\left(1 - \frac{r^2}{b'^2}\right)\left(1 - \frac{r^2}{c'^2}\right) - l^2\left(1 - \frac{r^2}{a'^2}\right) - m^2\left(1 - \frac{r^2}{b'^2}\right) - n^2\left(1 - \frac{r^2}{c'^2}\right) + 2lmn = 0,$$

a cubic equation in r^2, the roots of which are the squares of the principal axes; and by a discussion of which the properties of the conjugate diameters may be deduced.—Vide Hymers' *Analytical Geometry*.

As another example find the values of x, y, z, which [54] may make the function

$$u = (x + 1)(y + 1)(z + 1) \text{ a maximum,}$$

x, y, z being connected by the equation

$$a^x . b^y . c^z = A.$$

Taking the logarithmic differentials of both equations,

$$0 = \frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1},$$

$$0 = dx \log a + dy \log b + dz \log c.$$

Multiply the second by $-\lambda$, add and equate to 0 the coefficients of each differential, then

$$\lambda \log a = \frac{1}{x+1}, \quad \lambda \log b = \frac{1}{y+1}, \quad \lambda \log c = \frac{1}{z+1}.$$

Multiply by $x + 1$, $y + 1$, $z + 1$, and add; then

$$\lambda (x \log a + y \log b + z \log c + \log a + \log b + \log c) = 3,$$

or

$$\lambda \log (Aabc) = 3.$$

Therefore substituting

$$x + 1 = \frac{\log (Aabc)}{3 \log a}, (y + 1) = \frac{\log (Aabc)}{3 \log b}, z + 1 = \frac{\log (Aabc)}{3 \log c};$$

therefore $u = (x + 1)(y + 1)(z + 1) = \dfrac{\{\log (Aabc)\}^3}{\log a^3 . \log b^3 . \log c^3}.$

A. S.

ON THE SOLUTION OF LINEAR EQUATIONS OF FINITE AND MIXED DIFFERENCES.

In the preceding Number the principle of the separation of the symbols of operation from those of quantity, was applied to the solution of Differential Equations. We shall here apply the same principle to equations of Finite and Mixed Differences; but, as the method is not very different from that previously delivered, there will be no necessity for dwelling long on it.

The general form of the linear equation of Finite Differences with constant coefficients, is

$$u_{x+n} + Au_{x+n-1} + \&c. + Ru_{x+1} + Su_x = X,$$

where A, B, C, &c. are constants, and X a function of x. By a known relation we can express the quantities u_{x+n}, u_{x+n-1}, &c. [55] in terms of u_x and its successive differences, so that the equation may be transformed into one which may be put under the form

$$f(\Delta)\, u_x = X;$$

which we might proceed to solve by means of the separation of the symbols. But it will be more convenient to proceed in the following manner. Since we have, generally,

$$u_{x+n} = (1 + \Delta)^n\, u_x,$$

we may consider $1 + \Delta$ as a separate symbol, subject to the same laws as d and Δ; the effect of which, when applied on u_x, is to convert u_x into u_{x+1}. If we represent, for the sake of shortness, $1 + \Delta$ by D, the equation will take the form

$$(D^n + AD^{n-1} + \&c. + RD + S)\, u_x = X,$$

or, as we may write it for convenience,

$$F(D)\, u_x = X.$$

Then, converting $F(D)$ into factors of the form $D - a$, a being one of the roots of the equation $F(z) = 0$, we can perform the inverse operation for each factor separately, and so arrive at the value of u_x. To do this more readily, we shall avail ourselves of a theorem similar to that given in page 25. This theorem is, that

$$(D - a)^n\, X = a^{n+x}\, \Delta^n\, (Xa^{-x}).$$

For if we expand the first side, it may be put under the form

$$a^n\, \{D^n a^{-n} - nD^{n-1} a^{-(n-1)} + \frac{n.n-1}{1.2} D^{n-2} a^{-(n-2)} - \&c.)\, X.$$

Now $\quad D^p a^{-x} = a^{-(x+p)}$, and $a^{-p} = a^x D^p a^{-x}$;

so that the expression may be put under the form

$$a^{n+x}\, (D^n D'^n - nD^{n-1} D'^{n-1} + \&c.)\, Xa^{-x},$$

where the accented letters refer to a^{-x} and the unaccented to X. This expression is equivalent to

$$a^{n+x}\, (DD' - 1)^n\, Xa^{-x}$$

$$= a^{n+x}\, (\Delta + \Delta' + \Delta\Delta')^n\, Xa^{-x} = a^{n+x} \Delta^n\, (Xa^{-x}),$$

by a theorem analogous to that of Leibnitz.

To apply this to the general equation $F(D)\, u_x = X$, suppose $F(D)$ to be resolved into binomial factors, such as $D - a$, so that it becomes

$$(D - a_1)\, (D - a_2) \ldots\ldots (D - a_n)\, u_x = X.$$

Then taking the inverse operation of $D - a$,

$$F'(D)\, u_x = (D - a_1)^{-1}\, X = a_1^{x-1}\, \Sigma\, (Xa_1^{-x}),$$

representing the product of the $n-1$ binomial factors by $F'(D)$, and omitting the arbitrary constant, as included in the sign of integration. Again, performing the inverse operation of $D - a_2$, we have

$$F''(D)\,u_x = a_2^{x-1}\,\Sigma\left\{a_1^{x-1}\,a_2^{-x}\,\Sigma(Xa_1^{-x})\right\} \qquad [56]$$

$$= \frac{a_2^x}{a_1 a_2}\,\Sigma\left\{\left(\frac{a_1}{a_2}\right)^x \Sigma(Xa_1^{-x})\right\},$$

and integrating this by parts,

$$F''(D)\,u_x = \frac{a_1^{x-1}}{a_1 - a_2}\,\Sigma(Xa_1^{-x}) + \frac{a_2^{x-1}}{a_2 - a_1}\,\Sigma(Xa_2^{-x}).$$

Proceeding in the same manner with the next factor,

$$F'''(D)\,u_x = \frac{a_3^{x-1}}{a_1 - a_2}\,\Sigma\left\{a_1^{x-1}\,a_3^{-x}\,\Sigma(Xa_1^{-x})\right\}$$

$$+ \frac{a_3^{x-1}}{a_2 - a_1}\,\Sigma\left\{a_2^{x-1}\,a_3^{-x}\,\Sigma(Xa_2^{-x})\right\}$$

$$= \frac{a_1^{x-1}}{(a_1 - a_2)(a_1 - a_3)}\,\Sigma(Xa_1^{-x}) + \frac{a_2^{x-1}}{(a_2 - a_1)(a_2 - a_3)}\,\Sigma(Xa_2^{-x})$$

$$+ \frac{a_3^{x-1}}{(a_3 - a_1)(a_3 - a_2)}\,\Sigma(Xa_3^{-x}),$$

after integration by parts and reduction. And in the same way we may proceed till all the binomial factors of operation are exhausted. It is obvious from this, how close an analogy this result bears with the corresponding one in Differential Equations.

If there be n equal roots, the theorem given above enables us to arrive directly at the solution. For in this case we should have

$$u_x = (D - a)^{-n}\,X = a^{x-n}\,\Sigma^n\,(Xa^{-x});$$

or introducing the arbitrary constants, which were before neglected,

$$u_x = a^{x-n}\,\Sigma^n\,(Xa^{-x}) + a^{x-n}\,(C + C_1 x + C_2 x^2 + \ldots\ldots + C_n x^n).$$

Linear equations of the first degree, with variable coefficients, may be easily converted into equations with constant coefficients, by a change analogous to the change of the independent variable in the Differential Calculus. For let

$$u_{x+1} + aP_x u_x = X$$

be the given equation, where P_x is a function of x. We may

assume $P_x = \frac{Q_{x+1}}{Q_x}$, so that dividing by Q_{x+1} the equation becomes

$$\frac{u_{x+1}}{Q_{x+1}} + a\frac{u_x}{Q_x} = \frac{X}{Q_x};$$

which may be solved as a linear equation in $\frac{u_x}{Q_x}$, from which, when found, we may determine u_x. The form of Q_{x+1} is evidently

$$P_1 P_2 \ldots\ldots P_x.$$

The same method may be sometimes applied to equations of a higher order. If, for instance, we have the equation

$$u_{x+2} + a\phi(x+1)\,u_{x+1} + b\phi(x)\,\phi(x+1)\,u_x = c.$$

[57] Let us represent the continued product

$$\phi(1).\phi(2)\cdots\phi(x+1)$$

by P_{x+1}. Then

$$\phi(x)\,\phi(x+1) = \frac{P_{x+1}}{P_{x-1}} \text{ and } \phi(x+1) = \frac{P_{x+1}}{P_x}.$$

Dividing by P_{x+1}, we get

$$\frac{u_{x+2}}{P_{x+1}} + a\frac{u_{x+1}}{P_x} + b\frac{u_x}{P_{x-1}} = \frac{C}{P_{x+1}},$$

which may be solved as an equation in $\frac{u_x}{P_{x-1}}$, and so the value of u_x determined.

It will be seen that the method here laid down of integrating equations of differences, possesses an advantage over the common one of not requiring any assumption of a form of solution; but this advantage is even more displayed in solving equations of mixed differences. In these the two symbols d and Δ are involved at the same time; but as they are the characteristics of independent operations, the same principles which were applied to each separately will hold when they are combined. Without discussing the general equations of mixed differences, we shall proceed to a few of the more interesting examples. Let us have the equation

$$\frac{d}{dx}\Delta y + a\frac{dy}{dx} + b\Delta y + aby = X.$$

By separating the symbols this may be put under the form

$$\left(\frac{d}{dx} + b\right)(\Delta + a)\,y = X.$$

We shall obtain a different result according as we operate first with the one or the other factor; taking then the differential factor first, we have

$$(\Delta + a)\, y = \varepsilon^{-bx} \int X \varepsilon^{bx} dx + C\varepsilon^{-bx}.$$

Now $\quad \Delta + a = 1 + \Delta - (1 - a) = D - (1 - a).$

Therefore, performing the inverse operation by the theorem given above (p. 55), we obtain

$$y = (1 - a)^{x-1}\, \Sigma \{(1 - a)^{-x}\, \varepsilon^{-bx} \int (X \varepsilon^{bx} dx)\}$$
$$+ C (1 - a)^{x-1}\, \Sigma\, \{\varepsilon^{-bx} (1 - a)^{-x}\} + (1 - a)^{x}\, \phi\, (\sin 2\pi x,\ \cos 2\pi x),$$

the last term being the complementary function arising from the operation $\{D - (1 - a)\}^{-1}$.

And performing the operation indicated in the second term, we have

$$y = (1 - a)^{x-1}\, \Sigma\, \{(1 - a)^{-x}\, \varepsilon^{-bx} \int (X \varepsilon^{bx}\, dx)\}$$
$$+\ C_1 \varepsilon^{-bx} + (1 - a)^{x}\, \phi\, (\sin 2\pi x,\ \cos 2\pi x).$$

If we begin with the other factor we have

$$\left(\frac{d}{dx} + b\right) y = (1 - a)^{x-1}\, \Sigma\, \{X\, (1 - a)^{-x}\} + (1 - a)^{x}\, \phi, \quad [58]$$

and $\quad y = \varepsilon^{-bx} \int dx\, [\varepsilon^{bx} (1 - a)^{x-1}\, \Sigma\, \{X\, (1 - a)^{-x}\}]$

$$+\ \varepsilon^{-bx} \int dx\, \{\varepsilon^{bx}\, (1 - a)^{x}\, \phi\} + C\varepsilon^{-bx},$$

where ϕ is put for ϕ (sin $2\pi x$, cos $2\pi x$).

The next example we shall take is one given by Paoli, where two variables are involved,

$$u_{x+1,y} - \frac{d}{dy}\, u_{x,y} = P_{x,y},$$

which may be put under the form

$$\left(D - \frac{d}{dy}\right) u_{x,y} = P_{x,y},$$

where the D refers to the x only.

Then following the same principle as that pursued by Mr. Greatheed in the *Philosophical Magazine* for September 1837, that is, considering the characteristic $\frac{d}{dy}$ as an independent quantity, we have

$$u_{x,y} = \left(\frac{d}{dy}\right)^{x-1} \Sigma \left(\frac{d}{dy}\right)^{-x} P_{x,y} + \left(\frac{d}{dy}\right)^{x} \phi\, (y),$$

an arbitrary function of y being substituted for a constant,

as the operations are partial. This may evidently be put under the form

$$u_{x,y} = \frac{d^x\left(\Sigma \int^{x+1} P_{x,y} dy^{x+1}\right)}{dy^x} + \frac{d^x \phi(y)}{dy^x}.$$

If $P_{x,y} = 0$, the result is the same as that given by Sir John Herschel, in p. 38 of his *Examples*. He likewise gives the equation

$$u_{x+2,y} - a\left(\frac{d}{dy}\right) u_{x+1,y} + b\left(\frac{d^2}{dy^2}\right) u_{x,y} = 0,$$

which may be put under the form

$$\left(D^2 - aD\frac{d}{dy} + b\frac{d^2}{dy^2}\right) u_{x,y} = 0,$$

where D affects x only. Or, putting it into the form of factors,

$$\left(D - m\frac{d}{dy}\right)\left(D - n\frac{d}{dy}\right) u_{x,y} = 0,$$

where m, n are the roots of the equation

$$z^2 - az + b = 0.$$

Then $$\left(D - n\frac{d}{dy}\right) u_{x,y} = m^{x-1}\,.\left(\frac{d}{dy}\right)^{x-1}.\,\phi(y),$$

[59] and

$$u_{x,y} = n^{x-1}\,.\left(\frac{d}{dy}\right)^{x-1} \Sigma\left\{m^{x-1}\,.\,n^{-x}\,.\left(\frac{d}{dy}\right)^{x-1}.\left(\frac{d}{dy}\right)^{-x}.\,\phi(y)\right\}$$
$$+ n^{x-1}\left(\frac{d}{dy}\right)^{x-1} \phi_1(y);$$

whence $$u_{x,y} = \frac{m^{x-1}\, d^{x-1}\phi_2(y)}{dy^{x-1}} + \frac{n^{x-1}\, d^{x-1}\phi_1(y)}{dy^{x-1}},$$

as the functions of y are quite arbitrary.

In the same manner we might integrate this equation, if the second side were some function of x; and also equations of the form

$$\Delta^2 u_{x,y} - a\Delta\frac{d}{dy} u_{x,y} + b\frac{d^2}{dy^2} u_{x,y} = X,$$

where Δ affects x only. But it is needless to dwell on these, and we shall therefore proceed to the integration of an equation, in which the advantages of this method are very conspicuous. Mr. Airy, on the hypothesis that the distances of the particles of the luminiferous ether are not infinitely

small compared with their sphere of action, has given the following equation for determining the disturbance of a particle:

$$\frac{d^2}{dt^2}u_{x,t}=\frac{a^2}{h^2}\Delta^2 u_{x-h,t},$$

where Δ affects x only. This may be put under the form

$$\left(\frac{d^2}{dt^2}-\frac{a^2}{h^2}\frac{\Delta^2}{1+\Delta}\right)u_{x,t}=0.$$

Following the same method as before, by splitting the operating factor into two, and integrating with each separately, we obtain

$$u_{x,t}=\varepsilon^{\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\phi(x)+\varepsilon^{-\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are arbitrary functions of x. To reduce this to a more intelligible form, we may avail ourselves of Fourier's formula for the transformation of functions. From it we have

$$\pi\phi(x)=\int_{-\infty}^{+\infty}d\alpha\,\phi(\alpha)\int_0^{\infty}dp\cos p(x-\alpha).$$

Substituting this form,

$$\pi u_{x,t}=\int_{-\infty}^{+\infty}d\alpha\,\phi(\alpha)\int_0^{\infty}dp\,\varepsilon^{\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\cos p(x-\alpha)$$
$$+\int_{-\infty}^{+\infty}d\alpha\,\psi(\alpha)\int_0^{\infty}dp\,\varepsilon^{-\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\cos p(x-\alpha).$$

We must now expand the exponential, and operate with each term separately on the cosine. When expanded it becomes

$$\left(1+\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}+\frac{a^2h}{h^2 1.2}t^2\frac{\Delta^2}{1+\Delta}+\&c.\right)\cos p(x-\alpha).$$

Now [60]

$$\frac{\Delta}{\sqrt{(1+\Delta)}}f(x)=\left\{\sqrt{(1+\Delta)}-\frac{1}{\sqrt{(1+\Delta)}}\right\}f(x)$$
$$=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right);$$

therefore $$\frac{\Delta}{\sqrt{(1+\Delta)}}\cos p(x-\alpha)=-2\sin\frac{ph}{2}\sin p(x-\alpha),$$

$$\frac{\Delta^2}{1+\Delta}\cos p(x-\alpha)=-\left(2\sin\frac{ph}{2}\right)^2\cos p(x-\alpha),$$

and so on for every repetition of the operation, so that the series becomes

$$\left\{1 - \frac{a^2.h^2}{h^2.1.2}\left(2\sin\frac{ph}{2}\right)^2 + \frac{a^4.t^4}{h^4.1.2.3.4}\left(2\sin\frac{ph}{2}\right)^4 - \&c.\right\} \times$$
$$\times \cos p(x-a)$$
$$-\left\{\frac{a}{h}\frac{t}{1}\left(2\sin\frac{ph}{2}\right) - \frac{a^3.t^3}{h^3.1.2.3}\left(2\sin\frac{ph}{2}\right)^3 + \&c.\right\}\sin p(x-a)$$
$$= \cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-a) - \sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-a).$$

Similarly for the other term we shall obtain

$$\cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-a) + \sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-a):$$

and if these expressions be substituted in the expression for $u_{x,t}$, we shall obtain a result free from the symbol Δ: but a slight change in the form of the function will reduce it to a somewhat better form; for if we make

$$f(a) = \psi(a) + \phi(a) \text{ and } F(a) = \psi(a) - \phi(a),$$

the expression becomes

$$\pi u_{x,t} = \int_{-\infty}^{+\infty} da f(a) \int_0^\infty dp \cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-a)$$
$$+ \int_{-\infty}^{+\infty} da F(a) \int_0^\infty dp \sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-a),$$

which is the general solution of the equation. The form of the arbitrary functions are easily determined from the initial circumstances. For if in the last expression, or in the original integral, we make $t = 0$, we find

$$u_{x,0} = f(x),$$

which determines the form of the function f from the initial state of the particles. If we differentiate the expression for $u_{x,t}$ with regard to t, and then make $t = 0$, there results

$$\frac{du_{x,0}}{dt} = \frac{a}{h}\left\{F\left(x - \frac{h}{2}\right) - F\left(x + \frac{h}{2}\right)\right\},$$

which determines the form of the function F from the initial velocity.

[61] Mr. Airy has found a particular integral of this equation by assuming as a form of solution

$$u = A\sin\frac{2\pi}{\lambda}(vt - x + a),$$

and then determining v. But his result may be arrived at merely by assuming the form of the arbitrary function to be that used by Taylor in the solution of the problem of vibrating chords, and which is usually taken as the form of the function in the undulatory theory, and is for that reason well adapted for comparing the results. Let then

$$\phi(x) = \sin \frac{2\pi}{\lambda}(x + a), \quad \psi(x) = 0.$$

Effecting the operations indicated in the original integral by the same method as that used in reducing Fourier's function, we obtain

$$u = \sin \frac{2\pi}{\lambda}\left\{x + \frac{a\lambda}{\pi}\sin\left(\frac{\pi h}{\lambda}\right)t + a\right\}.$$

This expression gives us for the velocity of propagation of the wave, which is the coefficient of t,

$$v = \frac{a\lambda}{\pi}\sin\frac{\pi h}{\lambda},$$

which is no longer independent of λ, and therefore the velocity will be different for different values of λ; and as the index of refraction is inversely as the velocity, it will be nearly inversely as λ, and will thus be greater for violet than red light. This result then, as far as it goes, accounts for the phenomena of dispersion.

D. F. G.

Trin. Coll.

DEMONSTRATIONS OF SOME PROPERTIES OF THE CONIC SECTIONS.

1. THE position of the circle of curvature at any point of a Conic Section, may be readily determined by the following construction. Describe a circle touching the curve at the given point and cutting it in two others; then the chord in the conic section which passes through the given point and is parallel to the line joining the two points of section, is a chord of the circle of curvature at the given point.

Taking the origin at the point in the curve, the equation to the conic section is

$$y^2 + Bxy + Cx^2 + Dy + Ex = 0.$$

If we make the normal the axis of x and the tangent [62]

that of y, when $x = 0$ the two values of y must also $= 0$, and therefore $D = 0$, which reduces the equation to

$$y^2 + Bxy + Cx^2 + Ex = 0.$$

The equation to a circle touching the curve at the given point, that is, being also a tangent to the axis of y at that point, is

$$y^2 + x^2 + E'x = 0.$$

At the intersection of the curves we may combine the equations in any manner; subtracting them, we have

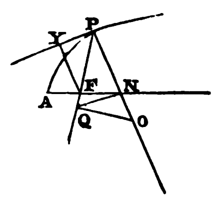
$$Bxy + (C - 1)\,x^2 + (E - E')\,x = 0,$$

which gives

$$x = 0, \quad By + (C - 1)\,x + E - E' = 0.$$

The first of these equations gives the origin, the second gives a relation between the coordinates of each of the points of section; and as it is linear it is the equation to the straight line joining them. Now the angle which this line makes with the axis of x is determined by the ratio $\dfrac{1 - C}{B}$; and as this is independent of the circle, the line will remain parallel to one position, as the circle varies in size. But when the circle becomes the circle of curvature, one of the points of intersection coincides with the point of contact, and the line joining the points of intersection, which remains parallel to one position, must pass through the point of contact. Therefore, if from this point a line be drawn parallel to a known position of the line of intersection, it will pass through the point in which the circle of curvature cuts the curve, and will thus be a chord both in the curve itself and in the circle of curvature. Knowing now one chord of the circle of curvature, and the position of its diameter, which coincides with that of the normal, we can determine the circle altogether.

2. The following is another and a very curious method of determining the centre of curvature in the Conic Sections. It was first given by Keill, but seems to have been rather strangely neglected by the subsequent writers on this subject.



From any point P in the curve draw the normal PN cutting the axis in N; at the point N draw NQ perpendicular to the normal, and meeting the focal chord through P in Q. From Q draw QO perpendicular to the focal chord and meeting the normal in O; then O is the centre of the circle of curvature

at the point P. Draw FY from the focus perpendicular to the tangent, and let

$$FP = r, \quad FY = p, \quad PN = N.$$

In the right-angled triangle PQO, we have plainly

$$PO \, . \, PN = PQ^2.$$

But from the similar triangles QPN, FPY,

$$PQ \, . \, FY = FP \, . \, PN;$$

$$\text{therefore } PQ^2 = N^2 \frac{r^2}{p^2}, \text{ and } PO = N \frac{r^2}{p^2}. \qquad [63]$$

Also in a conic section we have, if R be the radius of curvature,

$$R = \frac{N^3}{m^2} = \text{also } \frac{mr^3}{p^3},$$

where m is half the *latus rectum*.

From which we have $\frac{r^2}{p^2} = \frac{N^2}{m^2}$, and therefore

$$PO = \frac{N^3}{m^2} = R,$$

and O is the centre of the circle of curvature at P.

3. In the *Cambridge Transactions*, vol. 3, Mr. Morton has demonstrated a number of curious properties of the Conic Sections in relation to the generating cone; but he does not seem to have noticed the following one. If a sphere be described round the vertex of a cone as centre, the *latera recta* of all sections of the cone made by planes touching the sphere are equal. Taking the vertex of the cone as the origin, and the axis of the cone as the axis of x, the equation to the cone will be

$$x^2 + y^2 = m^2 z^2.$$

And if we suppose the cutting plane to be perpendicular to the plane of xz, its equation will be

$$z \cos \alpha + x \sin \alpha = r;$$

where r is the radius of the sphere, and α the angle which the perpendicular from the origin on the plane makes with the axis of z. Eliminating z between the two equations, we get

$$x^2 (\cos^2 \alpha - m^2 \sin^2 \alpha) + y^2 \cos^2 \alpha + 2m^2 rx \sin \alpha = m^2 r^2,$$

which is the equation to the projection of the section on the plane of xy. This equation, which is that to an ellipse, is not referred to its centre, but if we so refer it, it becomes

$$x^2 (\cos^2 \alpha - m^2 \sin^2 \alpha)^2 + y^2 \cos^2 \alpha \, (\cos^2 \alpha - m^2 \sin^2 \alpha) = m^2 r^2 . \cos^2 \alpha.$$

Now, if $a,' b'$ be the axes of the projection,

$$a'^2 = \frac{m^2 r^2 \cos^2 a}{(\cos^2 a - m^2 \sin^2 a)^2}, \quad b'^2 = \frac{m^2 r^2}{\cos^2 a - m^2 \sin^2 a}.$$

If a, b be the axes of the section, as the cutting plane is perpendicular to the plane of xz, and makes an angle α with the plane of xy,

$$a' = a \cos \alpha, \quad b' = b.$$

And for the *latus rectum*,

$$\frac{2b^2}{a} = 2mr,$$

[64] which being independent of α, is the same for all sections for which r is the same; that is, for all those which are made by planes touching the sphere.

From this it appears, that the *latus rectum* is equal to the diameter of the sphere multiplied by the tangent of half the vertical angle of the cone.

D. F. G.

ON THE MOON'S ANNUAL AND SECULAR INEQUALITIES.

THE following is a simple method of obtaining the expressions for the Moon's annual and secular equations, and which may be applied to determine those parts of the Moon's inequalities in longitude, which depend on the central disturbing force, and are independent of the eccentricity of the Moon's orbit and the square of the disturbing force.

In this investigation the Moon is supposed to be moving in a circle, the radius of which is variable. The equable description of areas is independent of the magnitude of the central force. Hence

$$h = (\mu a)^{\frac{1}{2}} \text{ is constant, } \mu \text{ and } a \text{ varying.}$$

Let $\delta\mu$, δa represent the *finite* variations of μ and a at any time. We suppose their squares to be neglected in this investigation, and hence they may be treated as if they were infinitesimal.

Differentiating the logarithm of the expression for h,

$$0 = \frac{\delta\mu}{\mu} + \frac{\delta a}{a} \dots\dots\dots\dots (1);$$

also $$\frac{d\theta}{dt} = \sqrt{\left(\frac{\mu}{a^3}\right)};$$

therefore $$\frac{\delta d\theta}{d\theta} = \tfrac{1}{2} \cdot \frac{\delta\mu}{\mu} - \tfrac{3}{2} \cdot \frac{\delta a}{a} \ldots\ldots\ldots\ldots (2);$$

and substituting the value of $\frac{\delta a}{a}$ from equation (1), and putting $d\delta\theta$ for $(\delta d\theta)$,

$$\frac{d\delta\theta}{d\theta} = 2\,\frac{\delta\mu}{\mu},$$

$$\delta\theta = 2\int\frac{\delta\mu}{\mu}\,d\theta \ldots\ldots\ldots\ldots\ldots (3),$$

which will give the Moon's inequality in longitude.

The central force is by hypothesis $\frac{\mu + \delta\mu}{a^2}$; in the [65] case of the Moon disturbed by the Sun, this is equal to $\frac{\mu}{a^2} - \frac{1}{2}\frac{m'a}{r'^3}$, where m', r' are the Sun's mass and distance.

Hence $$\frac{\delta\mu}{\mu} = -\tfrac{1}{2}\frac{m'a^3}{\mu r'^3}.$$

As the Sun describes an elliptic path about the Earth,

$$\frac{1}{r'} = \frac{1 + e'\cos(\theta' - \omega')}{a'(1 - e'^2)}.$$

Let m be the ratio of mean motions of the Sun and Moon, β the Sun's longitude when the Moon's longitude is 0,

$$\theta' = m\theta + \beta + \text{terms multiplied by } e'.$$

Hence, as far as the first power of e',

$$\frac{1}{r'} = \frac{1 + e'\cos(m\theta + \beta - \omega')}{a'};$$

substituting this value, and putting m^2 for $\frac{m'}{\mu} \cdot \frac{a^3}{a'^3}$,

$$\frac{\delta\mu}{\mu} = -\frac{m^2}{2} - \frac{3}{2}m^2e'\cos(m\theta + \beta - \omega').$$

The effect of the first term of this expression is to increase the Moon's periodic time by a constant quantity, which is in fact included in the mean motion determined from observation. The second term will give the inequality depending on the place of the Sun and the eccentricity of its orbit. Therefore, confining $\delta\theta$ to this part,

$$\begin{aligned}\delta\theta &= -\,3m^2e' \int \cos\,(m\theta + \beta - \omega')\,d\theta,\\ &= -\,3me' \sin\,(m\theta + \beta - \omega'),\\ &= -\,3me' \sin\,(\text{Sun's mean anomaly})\,;\end{aligned}$$

which is the expression for the annual equation. Vide *Airy*, Lunar Theory, No. 59.

The effect of that part of the Sun's attraction which is independent of the Moon's place in her orbit, is to diminish the Earth's attraction by the quantity

$$\tfrac{1}{2}\,\frac{m'a}{r'^3}\,.$$

This depends on the Sun's position in its orbit.

Its mean value will be found by adding its values taken at every instant during a complete revolution of the Sun, *i. e.* a year, and dividing by the number of instants; this amounts to integrating the quantity, multiplied by the element of time, for a year, and dividing by the length of a year.

[66] Let T represent the length of a year.

The mean value of $\tfrac{1}{2}\,\dfrac{m'a}{r'^3} = \dfrac{1}{T}\cdot\tfrac{1}{2}\,m'a\displaystyle\int_0^T \frac{dt}{r'^3}$,

$$r'^2 d\theta' = \sqrt{\{m'a'\,(1 - e'^2)\}}\;dt,$$

$$T = 2\pi a'^{\frac{3}{2}}\,m'^{-\frac{1}{2}}.$$

Hence mean value

$$= \frac{m'a}{2}\cdot\frac{\sqrt{m'}}{2\pi a'^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{m'a'\,(1 - e'^2)}}\cdot\int_0^{2\pi}\frac{d\theta'}{r'}\,.$$

$$= \frac{1}{4\pi}\cdot\frac{m'a}{a'^2}\cdot\frac{1}{\sqrt{(1 - e'^2)}}\int_0^{2\pi}\frac{1 + e'\cos\,(\theta' - \omega')}{a'\,(1 - e'^2)}\,d\theta',$$

$$= \tfrac{1}{2}\cdot\frac{m'a}{a'^3}\cdot\frac{1}{(1 - e'^2)^{\frac{3}{2}}},$$

$$= \tfrac{1}{2}\,\frac{m^2}{(1 - e'^2)^{\frac{3}{2}}}\cdot\frac{\mu}{a^2}\,.$$

Neither the Sun's mean distance, nor the length of the year which depends on it, alter sensibly. But in consequence of the action of the planets, the eccentricity of its orbit e' has

been constantly diminishing for centuries. The expression we have just found shows, that in consequence of this the value of the Sun's disturbing force is constantly diminishing, or the Earth's attraction increasing; and hence arises that inequality celebrated in the history of Astronomy as the secular acceleration of the Moon's mean motion.

To find its amount, suppose E the eccentricity of the Sun's orbit at any epoch,

$$2\frac{\delta\mu}{\mu} = -\frac{m^2}{(1-e'^2)^{\frac{3}{2}}},$$
$$= -m^2 - \tfrac{3}{2}m^2e'^2 + \&c.$$
$$= -m^2 - \tfrac{3}{2}m^2E^2 + \tfrac{3}{2}m^2(E^2 - e'^2) + \&c.$$

The effect of the first two terms is included in the Moon's mean motion, as determined by observation at the epoch in question: the inequality in longitude at any other time, depending on the diminution of e', will be

$$\delta\theta = \tfrac{3}{2}m^2 \int (E^2 - e'^2)\, d\theta,$$

which is the equation given in the *Mécanique Céleste*, liv. vii. No. 15. [67]

Let E' be the annual diminution of e', and let the year be taken as the unit of time;

$$e' = E - E't + \&c.; \quad \therefore\ E^2 - e'^2 = 2EE't \text{ nearly},$$
$$md\theta = 2\pi dt; \text{ therefore}$$
$$\delta\theta = 3m\pi EE' \int 2t\,dt,$$
$$= 3m\pi EE't^2,$$
$$= 3m\pi(\text{ecc}^{\text{y}}.\text{ of Sun's orb.})(\text{ann.var.of ecc}^{\text{y}}.)(\text{sq. of n}^{\text{o}}.\text{ of yrs.}).$$

A. S.

ON THE EXPANSION OF A FUNCTION OF A BINOMIAL.

Though Taylor's theorem naturally presents itself at the very entrance to the Differential Calculus, many points relating to it are imperfectly understood even by the best mathematicians. It does not seem to have occurred to any one, that the particular expansions of $f(a + h)$ in fractional or negative powers of h, to which we are obliged to have recourse where Taylor's series fails when x becomes a, are derivable from another general expansion of $f(x + h)$. This we shall show

to be the case. It seems, indeed, that many persons, misled by Lagrange's proposition, "the expansion of $f(x+h)$ cannot contain any fractional or negative powers of h unless x have a particular value," think that there is no such other general expansion; but the above proposition ought to be enounced thus, "that expansion of $f(x+h)$ *whose first term is* $f(x)$, cannot contain any fractional or negative powers of h, *and cannot fail* unless x have a particular value;" for the propositions, "the first term of the expansion is $f(x)$," and "the expansion cannot contain fractional or negative powers of h," are reciprocally dependent on one another, as will be seen. Professor Peacock, in his Report referred to in page 11 of our last Number, has employed general differentiation to express the general term of the expansion of $f(x+h)$; but in all other respects we believe that the contents of this Article are new.

2. Let the following general series be assumed for $f(x+h)$,

$$Ah^{\alpha} + A'h^{\alpha'} + A''h^{\alpha''} + \ldots\ldots\ldots,$$

where the indices α, α', α'' are supposed to be in order of magnitude either increasing or decreasing. Equating the first differential coefficients with respect to x and h,

[68]
$$\left.\begin{aligned} &\frac{dA}{dx}h^{\alpha} + \frac{dA'}{dx}h^{\alpha'} + \frac{dA''}{dx}h^{\alpha''} + \ldots\ldots \\ &= \alpha Ah^{\alpha-1} + \alpha'A'h^{\alpha'-1} + \alpha''A''h^{\alpha''-1} + \ldots\ldots \end{aligned}\right\} \ldots\ldots (1),$$

One way by which this equation may be satisfied is, by making $\alpha = 0$; this will lead to the known series of Taylor. But there is another way by which equation (1) may be satisfied; this is, by making $\frac{dA}{dx}$ equal to zero. Others of the differential coefficients may or may not vanish; we will suppose first, for the sake of simplicity, that they do not. Comparing the other terms with respect both to the indices and coefficients,

$$\alpha' = \alpha - 1, \quad \alpha'' = \alpha' - 1, \text{ \&c.}$$

$$\frac{dA'}{dx} = \alpha A, \quad \frac{dA''}{dx} = \alpha' A', \text{ \&c.:}$$

and since A is a constant $= C_0$,

$$A' = \int \alpha A dx = \frac{\alpha}{1} C_0 x + C_1,$$

$$A'' = \int (\alpha - 1) A' dx = \frac{\alpha(\alpha-1)}{1.2} C_0 x^2 + \frac{\alpha - 1}{1} C_1 x + C_2, \text{ \&c.}$$

Hence $\quad f(x+h) = C_0 h^a + \left(\frac{a}{1} C_0 x + C_1\right) h^{a-1}$

$$+ \left\{\frac{a(a-1)}{1.2} C_0 x^2 + \frac{a-1}{1} C_1 x + C_2\right\} h^{a-2} + \dots\dots\dots\dots \quad (2).$$

3. It is only in a particular form of $f(x)$ that $f(x+h)$ can be expanded in this series. To determine that form, put $x = 0$ in the last equation, whence

$$f(h) = C_0 h^a + C_1 h^{a-1} + C_2 h^{a-2} + \dots\dots\dots,$$

and changing h into x,

$$f(x) = C_0 x^a + C_1 x^{a-1} + C_2 x^{a-2} + \dots\dots\dots\dots \quad (3).$$

It follows then, that when $f(x)$ is of or can be expanded in the form (3), which includes all the simplest functions for which Taylor's series can fail, there exists another general development of $f(x+h)$ besides Taylor's series. In fact, the series (2) is the same as would result from changing x into $x+h$ in each term of (3) and developing each term by the binomial theorem in descending powers of h. It is evident that the series (2) cannot fail for any value of x. If all the indices of x in (3) be positive integers, the series (2) will be identical with Taylor's (which will in this case terminate) written in the contrary order.

4. The coefficients of the several powers of h in (2) may be expressed in terms of the general differential coefficients of $f(x)$ with respect to x. Putting $f(x) = u$, and using the formula (P) in page 20 of our last Number, we evidently have from (3), [69]

$$\frac{d^{a-r}u}{dx^{a-r}} = \frac{P(a)}{P(r)} C_0 x^r + \frac{P(a-1)}{P(r-1)} C_1 x^{r-1} + \dots$$

$$= \frac{a(a-1)\dots.(a-r+1)\,P(a-r)}{1.2\dots.r} C_0 x^r$$

$$+ \frac{(a-1)(a-2)\dots.(a-r+1)\,P(a-r)}{1.2\dots.(r-1)} C_1 x^{r-1} + \dots$$

As was proved in the article on General Differentiation, the terms in the preceding differential coefficient which contain negative powers of x (if there be any such) are finite, but, unless $a-r$ be a positive integer, $P(a-r)$ is infinite: hence in dividing by $P(a-r)$ the terms involving negative powers of x disappear, and we have

$$\frac{1}{P(a-r)} \frac{d^{a-r}u}{dx^{a-r}} = \frac{a(a-1)\dots(a-r+1)}{1.2\dots r} C_0 x^r$$

$$+ \frac{(a-1)(a-2)\dots(a-r+1)}{1.2\dots(r-1)} C_1 x^{r-1} + \dots + C_r,$$

which is identical with the coefficient of h^{a-r} in equation (2); hence when $f(x)$ is of the form (3), $f(x+h)$ may be expanded in the series

$$\frac{h^a}{P(a)}\frac{d^a u}{dx^a} + \frac{h^{a-1}}{P(a-1)}\frac{d^{a-1}u}{dx^{a-1}} + \frac{h^{a-2}}{P(a-2)}\frac{d^{a-2}u}{dx^{a-2}} + \ldots\ldots \quad (4).$$

When a is a positive integer, the terms involving negative powers of x would not disappear naturally from those differential coefficients whose index is positive, but they must be rejected, in order that the expansion may agree with (2).

5. The following examples may serve to explain the subject.

Let $u = x\sqrt{(a-x)}$: the expansion of this in descending powers of x is

$$\sqrt{(-1)}\left(x^{\frac{3}{2}} - \frac{1}{2}ax^{\frac{1}{2}} - \frac{1.1}{2.4}a^2x^{-\frac{1}{2}} - \frac{1.1.3}{2.4.6}a^3x^{-\frac{3}{2}} + \ldots\right);$$

which is of the form (3), and we find in this case $a = \frac{3}{2}$. Also from this expansion we obtain

$$\frac{d^{\frac{3}{2}}u}{dx^{\frac{3}{2}}} = \sqrt{(-1)}\left\{P\left(\tfrac{3}{2}\right) - \tfrac{1}{2}\frac{P\left(-\frac{1}{2}\right)}{P(-1)}ax^{-1} - \frac{1.1}{2.4}\frac{P\left(-\frac{1}{2}\right)}{P(-2)}a^2x^{-2} - \ldots\right\},$$

$$\frac{d^{\frac{1}{2}}u}{dx^{\frac{1}{2}}} = \sqrt{(-1)}\left\{\frac{P\left(\frac{3}{2}\right)}{P(1)}x - \tfrac{1}{2}P\left(\tfrac{1}{2}\right)a - \frac{1.1}{2.4}\frac{P\left(-\frac{1}{2}\right)}{P(-1)}a^2x^{-1} - \ldots\right\},$$

$$\frac{d^{-\frac{1}{2}}u}{dx^{-\frac{1}{2}}} = \sqrt{(-1)}\left\{\frac{P\left(\frac{3}{2}\right)}{P(2)}x^2 - \tfrac{1}{2}\frac{P\left(\frac{1}{2}\right)}{P(1)}ax - \frac{1.1}{2.4}P\left(-\tfrac{1}{2}\right)a^2 - \ldots\right\}.$$

&c.

[70] Substituting these values in the series (4), observing the relation

$$P(n) = nP(n-1), \text{ and that } P(-1) = \int x^{-1}dx = \frac{1}{0},$$

we find $(x+h)\sqrt{(a-x-h)} =$

$$\sqrt{(-1)}\left\{h^{\frac{3}{2}} + \left(\frac{3}{2}x - \frac{1}{2}a\right)h^{\frac{1}{2}} + \left(\frac{3.1}{2.4}x^2 - \frac{1}{2}\cdot\frac{1}{2}ax - \frac{1.1}{2.4}a^2\right)h^{-\frac{1}{2}} + \ldots\right\};$$

the $(p+1)^{\text{th}}$ term being

$$\sqrt{(-1)}\left\{\frac{3.1.(-1)\ldots(5-2p)}{2.4.6\ldots\ldots\ldots\ldots 2p}x^p - \frac{1}{2}\cdot\frac{1.(-1)\ldots(5-2p)}{2.4\ldots\ldots\ldots(2p-2)}ax^{p-1}\right.$$

$$\left. - \frac{1.1}{2.4}\cdot\frac{(-1)\ldots(5-2p)}{2\ldots(2p-4)}a^2x^{p-2} - \ldots - \frac{1.1\ldots(3-2p)}{2.4\ldots\ldots\ldots 2p}a^p\right\}h^{\frac{3-2p}{2}};$$

when x receives the particular value a, the terms after the second vanish, and the series becomes

$$\surd(-1)\,(h^{\frac{3}{2}} + ah^{\frac{1}{2}}).$$

6. Again, let u or $f(x) = \surd(x^2 - a^2)$. Since this is equal to

$$x - \frac{1}{2}\,\frac{a^2}{x} - \frac{1.1}{2.4}\,\frac{a^4}{x^3} - \ldots$$

we have $a = 1$,

$$\frac{du}{dx} = 1 + \frac{1}{2}\,\frac{a^2}{x^2} + \frac{1.1}{2.4}\,\frac{3a^4}{x^4} + \ldots.$$

$$\frac{d^{-1}u}{dx^{-1}} = \frac{x^2}{2} - \frac{1}{2}\,\frac{a^2x^0}{0} + \frac{1.1}{2.4}\,\frac{a^4}{2x^2} + \ldots.$$

$$\frac{d^{-2}u}{dx^{-2}} = \frac{x^3}{2.3} - \frac{1}{2}\,\frac{a^2x}{0.1} - \frac{1.1}{2.4}\,\frac{a^4}{2.1.x} - \ldots.$$

$$\frac{d^{-3}u}{dx^{-3}} = \frac{x^4}{2.3.4} - \frac{1}{2}\,\frac{a^2x^2}{0.1.2} - \frac{1.1}{2.4}\,\frac{a^4x^0}{2.1.0} + \ldots.$$

and since the values of $P(1)$, $P(0)$, $P(-1)$, $P(-2)$, $P(-3)$, are 1.1, $\frac{1}{0}$, $-\frac{1}{1.0}$, $\frac{1}{2.1.0}$ respectively, we have, rejecting negative powers of x from $\frac{du}{dx}$ and u,

$$f(x+h) = h + xh^0 - \frac{1}{2}\,a^2h^{-1} + \frac{1}{2}\,a^2xh^{-2} - \left(\frac{1}{2}\,a^2x^2 + \frac{1.1}{2.4}\,.a^4\right)h^{-3} + \ldots,$$

the $(p+1)^{\text{th}}$ term being (provided p be not less than 2)

$$(-1)^{p+1}\left(\frac{1}{2}\,a^2x^{p-2} + \frac{1.1}{2.4}\cdot\frac{(p-2)(p-3)}{1.2}\,a^4x^{p-4}\right.$$

$$\left. + \frac{1.1.3}{2.4.6}\cdot\frac{(p-2)(p-3)(p-4)(p-5)}{1\;.\;2\;.\;3\;.\;4}\,a^6x^{p-6} + \ldots\right).$$

This will agree with the expansion of $\surd(h^2 \pm 2ah)$, when [71]
x has the value $\pm a$.

7. Returning to equation (1), let us suppose that several of the terms in its first member, besides $\frac{dA}{dx}$, vanish, and that the indices of h in these terms are β, γ, &c. By comparing the other terms with respect to their indices and coefficients, we should obtain for the expression of $f(x+h)$

the sum of the series in (2), and of others similar to it with β, γ, &c. instead of a and different constants. Also, if we denote the second member of (3), without regard to the values of the constants, by $\phi(x, a)$, we should find that the form of $f(x)$, corresponding to the more general form of $f(x+h)$, is

$$\phi(x, a) + \phi(x, \beta) + \phi(x, \gamma), \text{ \&c.};$$

which form includes all functions that are commonly called Algebraic. We shall always suppose the difference of any two of the indices β, γ, &c. to be fractions; for if it was an integer, the two series would unite into one.

8. The more general equation corresponding to (4) will be

$$\left.\begin{aligned} f(x+h) = {} & \frac{h^a}{P(a)}\frac{d^a u}{dx^a} + \frac{h^{a-1}}{P(a-1)}\frac{d^{a-1}u}{dx^{a-1}} + \ldots \\ & + \frac{h^\beta}{P(\beta)}\frac{d^\beta u}{dx^\beta} + \frac{h^{\beta-1}}{P(\beta-1)}\frac{d^{\beta-1}u}{dx^{\beta-1}} + \ldots \\ & + \frac{h^\gamma}{P(\gamma)}\frac{d^\gamma u}{dx^\gamma} + \frac{h^\gamma}{P(\gamma-1)}\frac{d^{\gamma-1}u}{dx^{\gamma-1}} + \ldots \end{aligned}\right\} \ldots (5),$$

&c.,

with this provision, that whenever the index of differentiation is a positive integer, we reject from the differential coefficient terms involving fractional or negative powers of x.

For since $\beta - a$ is a fraction, the index of x in every term of $\frac{d^a}{dx^a}\phi(x, \beta)$ is fractional, and therefore its coefficient finite, so that all these terms will disappear, together with the negative powers of x in $\frac{d^a}{dx^a}\phi(x, a)$, on being divided by $P(a)$, which is infinite when a is not a positive integer. Hence

$$\frac{d^a u}{dx^a} = \frac{d^a}{dx^a}\phi(x, a),$$

and for the same reasons

$$\frac{d^{a-1}u}{dx^{a-1}} = \frac{d^{a-1}}{dx^{a-1}}\phi(x, a), \quad \frac{d^\beta u}{dx^\beta} = \frac{d^\beta}{dx^\beta}\phi(x, \beta), \text{ \&c.};$$

so that, after expanding each of the functions

$$\phi(x+h, a), \quad \phi(x+h, \beta), \text{ \&c.}$$

we obtain equation (5).

[72] 9. As a function which requires the more general ex-

pansion (5) we may take $(x^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{3}{2}}$. This expanded in descending powers of x by the binomial theorem is

$$x - \frac{3}{2}\, a^{\frac{2}{3}} x^{\frac{1}{3}} + \frac{3.1}{2.4}\, a^{\frac{4}{3}} x^{-\frac{1}{3}} + \frac{3.1.1}{2.4.6}\, a^2 x^{-1} + \ldots .$$

whence we see that the quantities α, β, γ, &c. are three in number, and that their values are 1, $\frac{1}{3}$, $-\frac{1}{3}$, for the rest of the indices in the last series differ from these by integers. The rest of the process is similar to that used in § 5 and § 6.

10. We have not yet obtained the most general expansion for $f(x + h)$. Suppose that $f(x) = \varepsilon^x \sqrt{(x - a)}$, and that we wish to expand $f(x + h)$ in a series that shall not fail when $x = a$. We may expand the factors ε^{x+h} and $\sqrt{(x + h - a)}$ separately, and multiply the series together. Now ε^{x+h} can only be expanded in ascending powers of h, and the other factor must be expanded in descending powers of h. By the multiplication of the two series, therefore, we should obtain one unlimited both ways, as long as x is not equal to a.

11. In order to obtain a general formula to suit such cases, assume for $f(x + h)$ a series unlimited both ways, namely,

$$\ldots . + Bh^b + Ah^a + Ah^\alpha + Bh^\beta + Ch^\gamma + \ldots .$$

where we suppose the indices to increase towards the right, and α to be the least positive index.

Equating the first differential coefficients with respect to x and h,

$$\ldots . + \frac{dB}{dx}\, h^b + \frac{dA}{dx}\, h^a + \frac{dA}{dx}\, h^\alpha + \frac{dB}{dx}\, h^\beta + \ldots .$$

$$= \ldots . + aAh^{a-1} + \alpha Ah^{\alpha-1} + \beta Bh^{\beta-1} + \gamma Ch^{\gamma-1} + \ldots .$$

If we suppose at first, for the sake of simplicity, that only one of the indices in the assumed series lies between 0 and 1, $\beta - 1$ must be the least positive index in the second member of the last equation, and by comparing the terms we obtain

$$b = a - 1, \quad a = \alpha - 1, \quad \alpha = \beta - 1, \quad \beta = \gamma - 1, \quad \text{\&c.}$$

$$\frac{dB}{dx} = aA, \quad \frac{dA}{dx} = \alpha A, \quad \frac{dA}{dx} = \beta B, \quad \frac{dB}{dx} = \gamma C, \quad \text{\&c.}$$

and therefore, expressing the other quantities in terms of A and α,

$$f(x + h) = \ldots\ \alpha(\alpha - 1)\, h^{\alpha-2} \int^2 A\, dx^2 + \alpha h^{\alpha-1} \int A\, dx + h^\alpha A$$
$$+ \frac{h^{\alpha+1}}{\alpha + 1}\, \frac{dA}{dx} + \frac{h^{\alpha+2}}{(\alpha + 1)(\alpha + 2)}\, \frac{d^2 A}{dx^2} + \ldots\ldots \quad (6).$$

12. It is easy to see what would be the consequence of supposing several of the indices in the assumed series to lie between 0 and 1, namely, that we should obtain for $f(x+h)$ the sum of several series similar to that in (6), but in which [73] A would be replaced by B, C, &c. successively, and a by β, γ, &c. *Such, then, is the most general expansion for* $f(x+h)$. It shews that $f(x+h)$, in its most general form, is the sum of several simpler functions of $x+h$, which may separately be expanded by the formula (6). It is enough to consider one of these functions at a time; we therefore return to that equation.

13. Let us examine the circumstances under which the series in (6) can become limited in either direction. The form of the series will not be altered if we suppose $h^a A$ to represent no longer the term with the least positive index, but the first or last term that does not vanish. Now all the terms to the left of that may be made to vanish by supposing $a=0$, but by no other supposition while the value of x is general. This leads us immediately to Taylor's series, which we thus see to be the only general expansion of $f(x+h)$ in ascending powers of h exclusively; and from there being no quantity in it left undetermined, such as a in (4), to be applicable to all functions.

The terms on the right of $h^a A$ may be made to vanish by supposing A a constant, and by that supposition only. The equation then becomes the same as (2).

14. It remains to determine the value of the quantity A in equation (6). This may be done in the following manner:

Let y be a function of x of the form (3), z any other function, and let y' and z' represent the values of these quantities when x becomes $x+h$. Then by (4)

$$y' = \frac{h^a}{P(a)}\frac{d^a y}{dx^a} + \frac{h^{a-1}}{P(a-1)}\frac{d^{a-1}y}{dx^{a-1}} + \frac{h^{a-2}}{P(a-2)}\frac{d^{a-2}y}{dx^{a-2}} + \dots$$

and, by Taylor's series,

$$z' = z + \frac{h}{1}\frac{dz}{dx} + \frac{h^2}{1.2}\frac{d^2z}{dx^2} + \dots.$$

The multiplication together of these two series will produce one unlimited in both directions, in which the coefficient of h^a will be

$$\frac{1}{P(a)}\frac{d^a y}{dx^a}z + \frac{1}{1.P(a-1)}\frac{d^{a-1}y}{dx^{a-1}}\frac{dz}{dx} + \frac{1}{1.2.P(a-2)}\frac{d^{a-2}y}{dx^{a-2}}\frac{d^2z}{dx^2} + \dots$$

$$= \frac{1}{P(a)}\left(\frac{d^a y}{dx^a}z + \frac{a}{1}\frac{d^{a-1}y}{dx^{a-1}}\frac{dz}{dx} + \frac{a(a-1)}{1.2}\frac{d^{a-2}y}{dx^{a-2}}\frac{d^2z}{dx^2} + \dots\right),$$

which, by Leibnitz's theorem generalized, is equal to

$$\frac{1}{P(a)} \frac{d^a . yz}{dx^a}.$$

In the same manner it may be seen that the coefficient of h^{a-p} is

$$\frac{1}{P(a-p)} \frac{d^{a-p}yz}{dx^{a-p}},$$

and that of h^{a+p}, [74]

$$\frac{1}{P(a+p)} \frac{d^{a+p}yz}{dx^{a+p}},$$

if it be remembered that all the differential coefficients of y whose indices are greater than a are finite. The terms of the series, therefore, follow the same law as those of the series in (6); and if we suppose $yz = u$, the two series must be identical, so that by determining the quantity A, (6) becomes

$$f(x+h)$$

$$= \ldots + \frac{h^{a-1}}{P(a-1)} \frac{d^{a-1}u}{dx^{a-1}} + \frac{h^a}{P(a)} \frac{d^a u}{dx^a} + \frac{h^{a+1}}{P(a+1)} \frac{d^{a+1}u}{dx^{a+1}} + \ldots (7).$$

When it requires the sum of several series such as that in (6) to express the value of $f(x+h)$, the determination of A and the other corresponding quantities presents some difficulties, so that we must leave it to a future occasion, and perhaps to an abler analyst.

S. S. G.

ON SOME ELEMENTARY PRINCIPLES IN THE APPLICATION OF ALGEBRA TO GEOMETRY.*

In pure Algebra, the independent existence of signs of affection is the immediate consequence of the universal applicability of its rules of operation. Thus, if the operation indicated in $a - b$ is always possible, whatever the relation between a and b may be, it follows that $a - (a + c)$, $a - a - c$, or (since $a - a = 0$) $- c$ must be a possible result of such an operation. The negative sign of affection, therefore, in Algebra, although it represents something not implied in the primary arithmetical definition of subtraction, is yet

* From a Correspondent.

intimately connected with that operation, and can have no meaning distinct from that of subtraction in any case where subtraction becomes possible, or, in other words, $a + (-b)$ and $a - (+b)$ are identically equivalent.

But in applied Algebra the case is somewhat different. There may be quantities of specific natures, which by their reciprocal relations afford us an obvious interpretation of the signs of affection, and so enable us to define Addition and Subtraction as *operations* in a manner no less general than is done in Algebra itself; that is, we may lay down rules for the performing those operations, which rules shall not be limited, as in Arithmetic, by the particular relations of the quantities, but shall be applicable in all cases. In such cases it is clear that the independent sign of affection is the
[75] foundation, and not, as in Algebra, the consequence of the rule of operation: it exists in the very nature of the quantity which is to be subject to the rule, and is the condition which makes such a rule possible. In such cases, therefore, we shall have two distinct meanings of the negative sign $-$, as it indicates a quantity of a particular kind, or an operation to be performed; the only condition connecting the two being, that the adoption of either interpretation must lead to such results as may be considered *algebraical* equivalents. There may exist any specific diversity, as in the case of *geometrical* parallelism; but such diversity, being of a kind not comprehended in the algebraic symbols, forms no obstacle to their *algebraical* equivalence.

In the application of Algebra to Geometry, it is to be borne in mind, that the subject represented by the algebraical symbols is not geometrical extension, but rather extension combined with direction. It is not the distance *between* two points, but the distance of one point *rightwards* or *leftwards* of another. It is, in short, theoretical *motion*.

Quantities of the former, pure geometrical kind, have no opposite relation of plus or minus to their two extremities: subtraction applied to such quantities is the expunging of a line, and when the magnitude has once been entirely expunged, the operation can be carried on no longer. The negative sign cannot be applied to distance alone, but to distance or progress in a given direction. A man who travels north, and afterwards travels the same or a greater distance south, has not traversed no space, nor a space less than zero, but his progress north is upon the whole either zero or negative. We say, therefore, that the quantities we have selected for the application of algebraical reasoning are essentially

positive and negative, independently of any rules of addition and subtraction; they are quantities which are not adequately defined until we know whether they are preceded by + or –, and we now proceed to inquire for necessary rules for the addition and subtraction of quantities of this description.

On Addition.

Let AB be a positive line drawn *from* A *towards* B, and represented by a or $+a$: required to add to it a positive line equal to $+b$.

Now all positive lines must be drawn in the same direction, say towards the right of the point of starting, or that point from which their description begins. To increase AB by a line drawn from left to right is impossible, except we start from the point B, or the *final* point in the motion by which AB was described. We must therefore *continue* the motion AB in the same direction, making $BC = b$, and AC, measured from A to C, and not from C to A, is the sum of the positive progressions $AB + BC$, or $a + b$.

If we wish to add a negative quantity, or $-b$, to AB, it is clear that the operation cannot be performed at A, the origin of AB, for a negative line drawn through A will increase the whole AB; and if such a line were drawn equal to a in length, [76] the result $a - a$, instead of being zero, would be a geometrical line equal twice AB in length: to *add* $-b$ to a, therefore, we must proceed in the same way as when we *add* $+b$ to a, i.e. we must draw BC from B in the negative direction, and as before the sum of the two is AC, a geometrical line, which is evidently positive or negative according as BC is $<$ or $>$ AC.

It appears, therefore, in both cases, that *addition* is an *operation* performed at B, the second extremity of AB.

On Subtraction.

From considerations such as the foregoing, we shall readily convince ourselves, that to subtract one positive line from another, or to perform the operation $a - (+b)$, can only be accomplished by drawing a line $AC = b$ from A, the origin of the minuend, and that the equivalent difference $AB - AC$ or $a - b$ is the distance CB *measured from the final point in the subtrahend to that of the minuend.*

This difference will obviously be negative when AC is $> AB$, being always expressed by CB.

If AC be drawn from A negatively, then CB is

$$a - (-b) = a + b,$$

and is properly represented by a line equal to the sum of the two parts a and b, and positive in its description.

It may be remarked, that this is an independent proof of the rule for the concurrence of signs, and one whose place is supplied by no other. For that rule, as proved in Arithmetic, is confined to the concurrence of a double operation, and in Algebra it is an arbitrary assumption; but the proof we have here obtained is the only one for the case of the concurrence of the signs of operation with those of affection.

With respect to the expressions $a + (-b)$ and $a - (+b)$, it is evident that we obtain for them two geometrical lines, specifically different, but always identical in magnitude and direction; and as magnitude and direction are the only elements which are denoted by our algebraical system of representation, these two lines cannot be algebraically distinguished.

The rules here laid down for Addition and Subtraction, verify the algebraical results no less beautifully when the quantities involved are impossible symbols, and the direction of the lines rectangular.

If $BC \perp$ to AB be drawn upwards or downwards through B, we shall have AC as before

$$= a + \{\pm b\sqrt{(-1)}\};$$

and if AC be similarly drawn through A, we shall have

$$CB = a - \{\pm b\sqrt{(-1)}\}.$$

So also, if AB be drawn in a direction inclined to the axis of a, and equal to $a + b\sqrt{(-1)}$, and if we draw through B,
[77] (and therefore add) a line

$$BC = c + d\sqrt{(-1)};$$

then AC, the sum of these two, will readily be seen to represent

$$(a + c) + (b + d)\sqrt{(-1)};$$

and if AB and AC represent

$$a + b\sqrt{(-1)} \text{ and } c + d\sqrt{(-1)}$$

respectively, we get

$$CB = a - c + (b - d)\sqrt{(-1)}.$$

D.

NOTES ON THE UNDULATORY THEORY OF LIGHT. NO. II.

THE theorems of Fresnel respecting the motion of light in crystallized media, were demonstrated by him by means of two surfaces, *the surface of elasticity* and *the ellipsoid of con-*

struction, which is related to the former by certain reciprocal relations. He was thus not only furnished with simple demonstrations of many important theorems, but was enabled to make use of the known properties of these surfaces for the purpose of suggesting corresponding properties of the rays and waves of light.

In the present state of the theory, it may be useful to show how these results may be obtained with as much or more simplicity from direct analysis; and without discarding the consideration of these surfaces, to use them rather to give clearness and connexion to our results than as methods of demonstration.

Since the publication of Fresnel's Memoir in the 7th volume of the *Memoirs of the Institute*, the chief addition which has been made to his results is the beautiful discovery of conical refraction by Sir W. Hamilton: this will be shown to flow naturally from the equations we get in investigating the equation to the wave surface.

The method of proving the existence of the circle of plane contact on the wave surface is due to Mr. Greatheed: the expressions for l, m, and n in terms of v_1, v_2, are given by Professor Sylvester in the *Philosophical Magazine*.

As in the preceding part, we shall take as coordinate axes the three axes of elasticity, and we shall assume that the forces excited by displacements equal to unity in those directions are a^2, b^2, c^2 respectively. We shall call a line or direction, the cosines of the inclinations of which to the axes are l, m, n, the *line* or *direction* $[l, m, n]$.

1. Let then $lx + my + nz = 0$ be the equation to the [78]
plane wave, and let (α, β, γ) be the direction of vibration; a displacement equal to unity will give rise to forces $a^2\alpha$, $b^2\beta$, $c^2\gamma$ parallel to the axes.

Resolve these forces along the displacement and perpendicular to it, and let the latter direction be determined by the cosines α', β', γ', so that (α, β, γ), $(\alpha', \beta', \gamma')$ being perpendicular to each other,

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$\alpha'^2 + \beta'^2 + \gamma'^2 = 1 \ldots\ldots\ldots\ldots\ldots (2).$$

To determine α', β', γ', we have another relation arising from this, that the resultant, the displacement, and the line $(\alpha', \beta', \gamma')$ being all in the same plane, must all be perpendicular to some line (f, g, h),

or

$$\left.\begin{aligned} f\alpha + g\beta + h\gamma &= 0 \\ f\alpha' + g\beta' + h\gamma' &= 0 \\ fa^2\alpha + gb^2\beta + hc^2\gamma &= 0 \end{aligned}\right\} \dots\dots\dots\dots (A).$$

Eliminating f, g, h by cross multiplication,

$$\frac{\alpha'}{\alpha}(b^2 - c^2) + \frac{\beta'}{\beta}(c^2 - a^2) + \frac{\gamma'}{\gamma}(a^2 - b^2) = 0 \dots\dots (3).$$

If $(\alpha', \beta', \gamma')$ be perpendicular to the plane,

$$\alpha' = l, \quad \beta' = m, \quad \gamma' = n,$$

and making these substitutions, the equations (1), (2), (3) will determine α, β, γ.

The part of the resultant resolved along the displacement is

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2;$$

and as the other part, which is perpendicular to the direction of displacement, is also perpendicular to the front of the wave, it may, as we have shown before, be neglected, we have a wave proceeding with a velocity

$$v = \sqrt{(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)} \dots\dots\dots\dots (4).$$

2. If we multiply together the equations (1) and (3), put under the form

$$\left.\begin{aligned} l\alpha + m\beta &= -n\gamma \\ \frac{l}{\alpha}(b^2 - c^2) + \frac{m}{\beta}(c^2 - a^2) &= -\frac{n}{\gamma}(a^2 - b^2) \end{aligned}\right\} \dots\dots (B),$$

we get the equation

$$lm\left\{\frac{\beta}{\alpha}(b^2 - c^2) + \frac{\alpha}{\beta}(c^2 - a^2)\right\} + \&c. = 0 \dots\dots (5),$$

a quadratic in $\frac{\alpha}{\beta}$, the last term of which, $\frac{b^2 - c^2}{c^2 - a^2}$, is negative, and therefore the roots are real; the same is the case in the equation we should find for $\frac{\gamma}{\beta}$: and thus the above equations give two real directions (α, β, γ).

[79] Let α_1, α_2; β_1, β_2; γ_1, γ_2, be the roots

$$\frac{\alpha_1\alpha_2}{\beta_1\beta_2} = \frac{b^2 - c^2}{c^2 - a^2},$$

or

$$\frac{\alpha_1\alpha_2}{b^2 - c^2} = \frac{\beta_1\beta_2}{c^2 - a^2} = \frac{\gamma_1\gamma_2}{a^2 - b^2};$$

hence $$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0 \text{ .... } (6),$$

and the two directions are perpendicular to each other.*

Whatever then be the nature of the vibration of the particles in the plane wave, we may resolve it in these two directions, and we shall have two waves, whose vibrations are perpendicular to each other, and proceed with different velocities v_1 and v_2, determined from equation (4); that is to say, *any plane disturbance will produce two plane waves polarized in planes at right angles to each other, and of different refrangibilities.*

3. v_1, v_2 are the roots of a quadratic equation, which may easily be found from the equations (A) by substituting l, m, n for α', β', γ'. The method is the same as that in page 7 of our last Number, and the result is the same, namely,

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0 \text{ .... } (7).$$

4. Having determined the velocities of propagation of a plane disturbance, the form of the wave surface, or that surface into which a disturbance at any point will diverge, may be found.

If we suppose the disturbance to be caused by an infinite number of plane waves passing through the origin, each too faint to cause any impression on the senses of itself; then each of these waves will be propagated with the velocities we have just determined, and the form of the sensible disturbance at any instant will be the locus of their ultimate intersections, or the surface touched by all the plane waves.

At the end of a unit of time the equation to one of these planes is

$$\left.\begin{aligned} &lx + my + nz = v \\ \text{with the relations}\quad &l^2 + m^2 + n^2 = 1 \\ &\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0 \end{aligned}\right\} \text{ .... } (C).$$

The process of elimination will be found in the *Cambridge Transactions*, Vol. vi. Part 1.

5. The following three equations are found after eliminating the differentials:

* For another demonstration of this theorem, see Vol. iii. p. 291.

[80]

$$\left.\begin{aligned} x &= \frac{lv}{a^2 - v^2}(a^2 - r^2) \\ y &= \frac{mv}{b^2 - v^2}(b^2 - r^2) \\ z &= \frac{nv}{c^2 - v^2}(c^2 - r^2) \end{aligned}\right\} \dots\dots\dots\dots (D),$$

in which $\quad r^2 = x^2 + y^2 + z^2,$

and the resulting equation to the wave surface is

$$\frac{a^2x^2}{a^2 - r^2} + \frac{b^2y^2}{b^2 - r^2} + \frac{c^2z^2}{c^2 - r^2} = 0 \dots\dots\dots (8).$$

If X, Y, Z be the angles which a radius vector makes with the coordinate axes, the polar equation is

$$\frac{a^2 \cos^2 X}{a^2 - r^2} + \frac{b^2 \cos^2 Y}{b^2 - r^2} + \frac{c^2 \cos^2 Z}{c^2 - r^2} = 0 \dots\dots (9).$$

6. We shall now discuss some of the equations we have obtained.

The two equations (B) will in general determine two of the quantities $\frac{\alpha}{\beta}$, $\frac{\gamma}{\beta}$, and the directions of the vibration lines will be definite; but if one of the quantities l, m, n, as for instance m, is zero, the two equations will become identical, provided

$$n^2(a^2 - b^2) - l^2(b^2 - c^2) = 0 \dots\dots\dots (10),$$

and thus the direction of the lines of vibration become indeterminate, if

$$l = \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad m = 0, \quad n = \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)} \dots\dots (E),$$

These values of l and n are real, provided a, b, c are in the order of magnitude, but not otherwise; and thus we should not get a similar property by making $l = 0$ or $n = 0$.

Putting $a^2 - b^2 + b^2 - v^2$ and $b^2 - v^2 - (b^2 - c^2)$ for $a^2 - v^2$ and $c^2 - v^2$ in equation (7), we get

$$\begin{aligned} &(b^2 - v^2)^2 + (b^2 - v^2) \times \\ &[n^2(a^2 - b^2) - l^2(b^2 - c^2) + m^2\{a^2 - b^2 - (b^2 - c^2)\}] \\ &\qquad - m^2(a^2 - b^2)(b^2 - c^2) = 0 \dots\dots\dots\dots (11), \end{aligned}$$

which on our supposition reduces to

$$(b^2 - v^2)^2 = 0.$$

Thus the two values of v^2 become equal to each other and to b^2. There is no separation of the plane wave; and therefore for the two directions determined by equations (E), and for these only, a single plane wave may be propagated. The normal to this plane is then a direction or *line of single normal velocity.*

7. If equation (9) be put under the form [81]

$$\frac{\cos^2 X}{\frac{1}{a^2}-\frac{1}{r^2}}+\frac{\cos^2 Y}{\frac{1}{b^2}-\frac{1}{r^2}}+\frac{\cos^2 Z}{\frac{1}{c^2}-\frac{1}{r^2}}=0,$$

it becomes exactly similar in form to equation (7), and we at once infer that the two values of r^2 become equal to each other and to b^2, provided

$$\cos X=\sqrt{\left(\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)},\quad \cos Y=0,\quad \cos Z=\sqrt{\left(\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)},$$

$$\text{or } \cos X=\frac{c}{b}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)},\ \cos Y=0,\ \cos Z=\frac{a}{b}\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}\ldots(F),$$

and the radius vector, which in general cuts the surface in two points on the same side of the origin, will in this direction cut it only in one. This then is a direction or *line of single ray velocity.*

8. These two directions, to which we have given Sir William Hamilton's names, *the lines of single normal velocity,* and *the lines of single ray velocity,* are nearly coincident in most crystals. They were not distinguished by observation before Fresnel's theory, but, from some remarkable optical properties which they possess, they received the common name of *optic axes.*

When a, b, and c are different, the equations (E) and (F) give two lines inclined at equal angles to the plane of xy, and crystals of this kind were called *biaxal;* if $b = a$ or $= c$, the two directions coincide, and such crystals were called *uniaxal.* Fresnel has given the name of optic axes sometimes to one set and sometimes to the other, and considerable confusion has arisen from the ambiguity. We shall use the terms *wave axis* and *ray axis;* names which prevent ambiguity, and which appear sufficiently to explain their own meaning.

9. By comparing equations (7) and (E) with equations (9) and (F), we see that, supposing the direction of a ray to be

the same as that of a wave normal, the equations for determining the wave velocity v and the directions of the wave axes, are changed into those for determining the ray velocity r and the directions of the ray axes, by substituting $\frac{1}{r}$ for v and for a, b, c, their reciprocals. And hence, that every relation between the wave velocity and the directions of the wave normal and wave axes, will give a corresponding relation for the ray.

If we have

$$\phi\,(l,\ m,\ n,\ v,\ a,\ b,\ c,\ \text{direction of wave axis}) = 0,$$

[82] we derive at once the corresponding relation

$$\phi\left(\cos X,\ \cos Y,\ \cos Z,\ \frac{1}{r},\ \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c},\ \text{direction of ray axis}\right) = 0 \ldots\ldots (G).$$

10. Equation (11) gives us the inclination of the wave normal to the axes in terms of the two wave velocities v_1, v_2; for as the last term is the product of the two roots of the equation $(b^2 - v_1^2)(b^2 - v_2^2) = -m^2(a^2 - b^2)(b^2 - c^2)$,

or
$$m = \sqrt{\left\{\frac{(b^2 - v_1^2)(v_2^2 - b^2)}{(a^2 - b^2)(b^2 - c^2)}\right\}};$$

similarly,

$$l = \sqrt{\left\{\frac{(a^2 - v_1^2)(a^2 - v_2^2)}{(a^2 - b^2)(a^2 - c^2)}\right\}},\quad n = \sqrt{\left\{\frac{(v_1^2 - c^2)(v_2^2 - c^2)}{(a^2 - c^2)(b^2 - c^2)}\right\}} \ldots (H).$$

From these expressions we may find ι, ι', the angles between the wave normal and the wave axes, in terms of the velocities v_1, v_2,

$$\cos\iota = l\sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)} + n\sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

Substituting the above values of l and n, this becomes

$$(a^2 - c^2)\cos\iota = \sqrt{\{(a^2 - v_1^2)(a^2 - v_2^2)\}} + \sqrt{\{(v_1^2 - c^2)(v_2^2 - c^2)\}};$$

squaring, putting $a^2 - c^2 - (v_1^2 - c^2)$ for $a^2 - v_1^2$, and making a similar substitution for $v_1^2 - c^2$, the equation reduces to

$$(a^2 - c^2)^2\cos^2\iota = (a^2 - c^2)^2$$
$$- [\sqrt{\{(a^2 - v_1^2)(v_2^2 - c^2)\}} - \sqrt{\{(a^2 - v_2^2)(v_1^2 - c^2)\}}]^2,$$
$$(a^2 - c^2)\sin\iota = \sqrt{\{(a^2 - v_1^2)(v_2^2 - c^2)\}} - \sqrt{\{(a^2 - v_2^2)(v_1^2 - c^2)\}}.$$

The expression for $\sin\iota'$ will only differ by the sign of one of the radicals; the product of the two expressions will therefore be the difference of the squares of the radicals, or

$$(a^2 - c^2)^2\sin\iota\sin\iota' = \pm(a^2 - c^2)(v_2^2 - v_1^2),$$

or
$$v_2^2 - v_1^2 = \pm(a^2 - c^2)\sin\iota\sin\iota' \ldots\ldots\ldots (12).$$

If ε, ε' be the angles between the ray and the ray axis, we deduce from the above at once

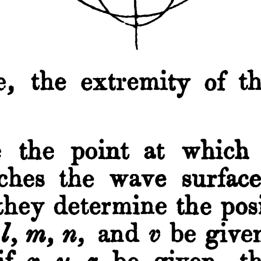
$$\frac{1}{r_2^2} - \frac{1}{r_1^2} = \pm\left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin \varepsilon \sin \varepsilon' \ldots\ldots (13),$$

or the difference of the reciprocals of the squares of the ray velocities is proportional to the product of the series of the inclinations of the ray to the ray axes. This law was discovered experimentally by Biot. In Airy's tract on the *Undulatory Theory*, No. 123, this property is inadvertently stated as belonging to the inclination of the ray to the *wave axes.*

11. We shall now proceed to consider the form of the wave surface given by equation (8).

The equations to its intersections with the coordinate [83] planes will consist, in each case, of a circle and ellipse. In the plane of xy the circle is contained within the ellipse; in the plane of yz the ellipse is contained within the circle; in the plane of xz the ellipse and circle cut each other, being given by the equation

$$(r^2 - b^2)\left(\frac{x^2}{c^2} + \frac{z^2}{a^2} - 1\right) = 0.$$



The point of intersection of the two curves is the extremity of the ray axis. The point where a common tangent to the two curves touches the circle is the extremity of the wave axis.

The figure represents a section of the surface by the plane of xz, D is the extremity of the ray axis, and F the point where the common tangent EF touches the circle, the extremity of the wave axis.

12. The equations (D) determine the point at which a tangent plane of given position touches the wave surface; or if the point of contact be given, they determine the position of the tangent plane: that is, if l, m, n, and v be given, they determine x, y, z; conversely, if x, y, z be given, the equations (D) with

$$lx + my + nz = v,$$

will determine l, m, n, v.

If for any values of l, m, n, v, one of these equations is satisfied independently of the values of x, y, and z, these

equations will determine, not a point, but a curve of contact with the tangent plane. Also, if for any values of x, y, z, one of the equations is satisfied independently of l, m, n, v, we shall get only two relations between l, m, n, and there will be an infinite number of tangent planes at that point, giving a cone of contact.

13. If $m = 0$, $v^2 = b^2$, the second equation is satisfied independently of x, y, z: this is the case we have before considered of the plane of single normal velocity.

The equations (D) are the equations of three spheres; hence, any two of them determine a circle: and thus it appears, that that plane wave which is propagated without separation, or along the wave axis, touches the wave surface in a circle, the position and magnitude of which we proceed to find.

We have, as before,

$$l = \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad n = \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)};$$

substituting these values, the two equations (D) become

$$x = \frac{b}{\sqrt{\{(a^2 - b^2)(a^2 - c^2)\}}}(a^2 - r^2),$$

$$z = \frac{b}{\sqrt{\{(a^2 - c^2)(b^2 - c^2)\}}}(r^2 - c^2);$$

[84] or putting $x^2 + y^2 + z^2$ for r^2,

$$x^2 + y^2 + z^2 + \frac{x}{b}\sqrt{\{(a^2 - b^2)(a^2 - c^2)\}} - a^2 = 0,$$

$$x^2 + y^2 + z^2 - \frac{z}{b}\sqrt{\{(a^2 - c^2)(b^2 - c^2)\}} - c^2 = 0.$$

If we eliminate z, we get for the projection on the plane of xy

$$\frac{a^2 - c^2}{b^2 - c^2}\left\{x - \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}\frac{b^2 + c^2}{2b}\right\}^2 + y^2 = \frac{(a^2 - b^2)(b^2 - c^2)}{4b^2},$$

the equation to an ellipse, of which the axis parallel to y is the radius of the circle of contact,

$$= \frac{\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}}{2b} \;\ldots\ldots\ldots\ldots (14).$$

This equation also gives one coordinate of the centre of the circle,

$$x = \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}\frac{b^2 + c^2}{2b} \;\ldots\ldots\ldots\ldots (15).$$

Similarly we should find

$$z = \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}\, \frac{a^2 + b^2}{2b} \text{ ....(16).}$$

And thus the position and magnitude of the circle of contact is completely determined.

From the double sign $\pm$ which these radicals involve, we see that there will be four such circles on the wave surface, one at each end of the two wave axes.

14. If $y = 0$, $r^2 = b^2$, the second equation is satisfied independently of l, m, n: this is the case at the extremity of the ray axis, where

$$x = c\sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad z = a\sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

Substituting these values, the first of equations (D) becomes

$$c\sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)} = \frac{lv}{a^2 - v^2}(a^2 - b^2),$$

or $$a^2 - v^2 = \frac{lv}{c}\sqrt{\{(a^2 - b^2)(a^2 - c^2)\}}:$$

similarly, $$v^2 - c^2 = \frac{nv}{a}\sqrt{\{(a^2 - c^2)(b^2 - c^2)\}};$$

and eliminating v, we get the equation

$$l^2(b^2 - c^2) + m^2(a^2 - c^2) + n^2(a^2 - b^2)$$
$$- ln\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}\,\frac{a^2 + c^2}{ac} = 0 \text{ .... (17);}$$

which, with the equation $l^2 + m^2 + n^2 = 1$, gives the relations between l, m, n. [85]

At this point there are an infinite number of tangent planes, whose ultimate intersections give a tangent cone. The equation found above, where x, y, z are put for l, m, n, is the equation to the surface described by a line always normal to these planes.

15. To find the equation to the tangent cone, call the above equation $u = 0$, $lx + my + nz = 0$ is a plane through the origin parallel to a tangent plane.

We have, therefore,

$$xdl + ydm + zdn = 0,$$
$$ldl + mdm + ndn = 0,$$
$$\frac{du}{dl}dl + \frac{du}{dm}dm + \frac{du}{dn}du = 0;$$

therefore
$$\frac{du}{dl} + \lambda l + \mu x = 0,$$
$$\frac{du}{dm} + \lambda m + \mu y = 0,$$
$$\frac{du}{dn} + \lambda n + \mu z = 0.$$

Multiplying by l, m, n, and adding, $\lambda = 0$. We have then to eliminate μ, l, m, n between

$$u = 0, \quad \frac{du}{dl} + \mu x = 0, \quad \frac{du}{dm} + \mu y = 0, \quad \frac{du}{dn} + \mu z = 0:$$

this is easily done, and the resulting equation

$$\frac{x^2}{b^2 - c^2} - \frac{(a^2 - c^2)\, y^2}{4a^2c^2} + \frac{z^2}{a^2 - b^2}$$
$$+ \frac{xz}{\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}} \frac{a^2 + c^2}{ac} = 0 \ldots\ldots (18)$$

is the equation to the tangent cone, the origin being at its vertex.

This is an elliptic cone: the axes of any section made by a plane perpendicular to its axis, may be found by transforming the coordinates so that the term involving xz may vanish.

The angle of the cone in the plane of xz may be found from either of the equations (17) or (18). If we make $m = 0$ in (17), and solve for $\frac{l}{n}$, we find

$$\frac{l}{n} = \sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)} \frac{a^2 + c^2 \pm (a^2 - c^2)}{2ac},$$

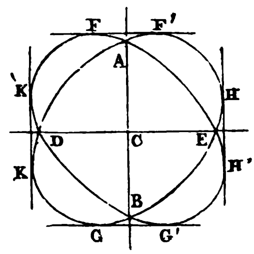
and the tangent of the angle sought

$$= \frac{\text{difference of roots}}{1 + \text{product of roots}}$$

$$= -\frac{\sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)} \cdot \frac{a^2 - c^2}{ac}}{1 + \frac{a^2 - b^2}{b^2 - c^2}}$$

$$= -\frac{\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}}{ac}.$$

At the extremity of the ray axis there is a *cusp*, or rather a *nodal point*, somewhat resembling the vertex of a very obtuse double cone. Perhaps a clearer conception of this surface may be formed by supposing an ellipse to revolve round a diameter AB, which is not one of the axes. At A and B there will be two nodal points, each having an infinite number of tangent planes forming two cones of contact; and the two tangent planes FF', GG' will touch the surface in two circles. At the points D and E there is a cusp or node of a different kind: here there will be only two different tangent planes, and the tangent planes HH', KK' will touch the surface in two points. The form of the surface at A and B will represent the form of the node of the wave surface very nearly, but not exactly, unless the ellipse be supposed to change its form while revolving.



From the existence of the four tangent cones at the extremities of the ray axes, and the four circles of plane contact corresponding to the wave of single normal velocity, Sir William Hamilton draws the following inferences, which have been fully confirmed by the experiments of Professor Lloyd.

1. That if a ray be refracted into the crystal so as to proceed along the ray axis, then, as the position of the tangent plane at the extremity of the axis is indeterminate, it will at emergence be refracted into an infinite number forming a cone, and giving *external conical refraction*.

2. That if a single ray be incident externally at an angle corresponding to the wave axis, it will be refracted internally into an infinite number forming a cone, and giving *internal conical refraction*. If the faces of the crystal be parallel, the rays will emerge parallel to the incident ray, forming an external cylinder of rays.

A. S.

SOLUTIONS OF SOME PROBLEMS IN TRANSVERSALS. [87]

THE name of Transversals was given by Carnot to lines considered in their relations of mutual intersection. Many of their properties are very curious, and form interesting

problems in Analytical Geometry, though it was not in this way that Carnot considered them. His method was, to proceed step by step from the more simple properties to the more complicated; but it seems better to consider each independently.

1. The following problem, under a slightly different form, was given in one of the Problem Papers for 1836.

If two lines, AB, CD intersect in O so that AB is bisected, and if the lines AC, BD meet, when produced, in E, and AD, BC in F, then the line EF is parallel to AB.

Take O as the origin, and AB, CD as the axes of x and y.

Let $\quad OB = a,\ OA = -a,\ OD = b,\ OC = -c.$

The equations to BD and AC are

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a} + \frac{y}{c} = -1.$$

At their intersection the equations may be combined in any manner; therefore, subtracting them,

$$y\left(\frac{1}{b} - \frac{1}{c}\right) = 2 \ldots\ldots\ldots\ldots\ldots (1).$$

The equations to BC and AD are

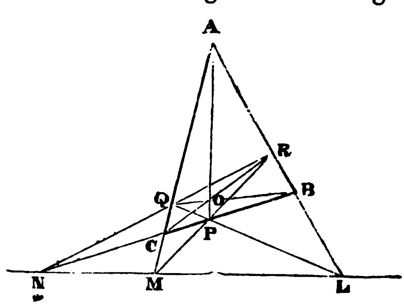
$$\frac{x}{a} - \frac{y}{c} = 1, \quad \frac{x}{a} - \frac{y}{b} = -1,$$

At their intersection, subtracting them,

$$y\left(\frac{1}{b} - \frac{1}{c}\right) = 2 \ldots\ldots\ldots\ldots\ldots (2),$$

which is the same as the equation (1), and therefore is the equation to the line passing through the points of intersection, that is, to EF; and from its form it is evident that it is parallel to the axis of x, that is, to AB.

2. If three lines be drawn from the angles of a triangle A, B, C, through one point O, and meeting the sides of the triangle in P, Q, R, the sides of the triangle P, Q, R will, when produced, meet those of A, B, C in three points, which are in the same straight line.



Take L, the point of intersection of AB and

PQ, as the origin, and these lines as the axes of x and y.

Let $LB = a_1$, $LR = a_2$, $LA = a_3$, $LP = b_1$, $LQ = b_2$. [88]

The equations to BC and QR are

$$\frac{x}{a_1} + \frac{y}{b_1} = 1, \quad \frac{x}{a^2} + \frac{y}{b_2} = 1.$$

At their intersection we have, by subtraction,

$$x\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \text{ .... (1).}$$

Again, the equations to AC and PR are

$$\frac{x}{a_3} + \frac{y}{b_2} = 1, \quad \frac{x}{a_2} + \frac{y}{b_1} = 1;$$

and at their intersection we have, by subtraction,

$$x\left(\frac{1}{a_2} - \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \text{ .... (2).}$$

Again, combining by addition the equations to BC and AC, we get at their point of intersection C,

$$x\left(\frac{1}{a_1} + \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} + \frac{1}{b_2}\right) = 2 \text{ .... (3).}$$

Also, the equations to AP and BQ are

$$\frac{x}{a_3} + \frac{y}{b_1} = 1, \quad \frac{x}{a_1} + \frac{y}{b_2} = 1;$$

and at their intersection we have, by addition,

$$x\left(\frac{1}{a_1} + \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} + \frac{1}{b_2}\right) = 2 \text{ .... (4),}$$

which is the same as (3), and therefore is the equation to COR, which passes through the two points of intersection.

If in this equation we make $y = 0$, we have $x = \dfrac{2a_1a_3}{a_1 + a_3}$, but in this case $x = LR = a_2$, therefore

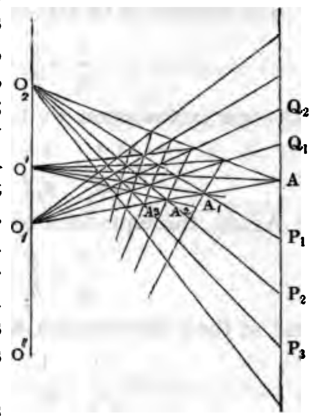
$$a_2 = \frac{2a_1a_3}{a_1 + a_3}, \text{ or } \frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_3};$$

therefore equations (1) and (2) coincide, and represent the line passing through M and N, and as from the form of the equation it evidently passes through the origin, the three points L, M, N are in one straight line.

In somewhat a similar manner it might be shown, that if three circles be touched, two and two, by pairs of tangents,

the points of intersection of these tangents are in one straight line. But we shall pass on to another problem.

3. If from any point A in a line of indefinite length equal distances AQ_1, Q_1Q_2, &c. be measured off, and other equal distances AP_1, P_1P_2, &c. in the opposite direction; and if Q_1, Q_2, Q_3, &c. be joined by straight lines with a point O_1, and
[89] P_1, P_2, P_3, &c. with a point O_2, O_1, and O_2 being situate in a line parallel to the given line; then the corresponding points of intersection of these lines will lie in straight lines forming two bundles of lines converging to two points O', O'', situate in O_1O_2.



Take A as the origin, AO_1 as the axis of x, AQ as the axis of y.

Let $\quad AO_1 = m, \;\; O_1O_2 = c, \;\; AQ_1 = a, \;\; AP_1 = b.$

Let one of the lines O_2P_n meet the axis of x in a point A_n; then by similar triangles it is easy to show that

$$\frac{1}{AA_n} = \frac{1}{m} + \frac{c}{mnb}.$$

The equation to O_1Q_r is

$$\frac{x}{m} + \frac{y}{ra} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

and the equation to O_2P_s is

$$\frac{x}{m} + \frac{cx}{msb} - \frac{y}{sb} = 1 \;\ldots\ldots\ldots\ldots\ldots\; (2).$$

Combining them by subtraction, after multiplying by r and s,

$$(s-r)\frac{x}{m} + \frac{cx}{mb} - y\left(\frac{1}{a} + \frac{1}{b}\right) = s - r \;\ldots\ldots\ldots\; (3),$$

which gives a relation between the coordinates of the point of intersection of the lines. We shall arrive at the same result for all pairs of lines for which $s - r$ is the same; consequently equation (3) will represent a line passing through the point of intersection of each pair. Now, let $x = m$, then

$$y = \frac{ac}{a+b},$$

which, being independent of r and s, will be the same for all the lines represented by (3) for which $s - r$ is different, and

therefore they all converge to one point, which it is easy to see lies between O_1 and O_2, as the value of y is less than that of c. If we make $x = 0$ in (3), we get

$$y = \frac{r - s}{\left(\frac{1}{a} + \frac{1}{b}\right)};$$

and as $s - r$ increases by unity, the distance between the points where the different lines diverging from O' cut the axis of y is half the harmonic mean between a and b.

If, instead of subtracting, we add equations (1) and (2), we get

$$(r + s)\frac{x}{m} + \frac{cx}{mb} + y\left(\frac{1}{a} - \frac{1}{b}\right) = r + s,$$

which, as before, may be shown to be the equation to a line passing through the intersections of all lines for which $r + s$ is the same. And if $r + s$ vary, we find, as before, that [90] the different lines converge to a point O'', such that

$$O_1O'' = -\frac{ac}{b - a}.$$

If $b > a$, O'' is situate in O_2O_1 produced. If $a = b$, O_1O'' becomes infinite, and the lines are then parallel to AQ. If $b < a$, the point O'' is above O_2. The distance between the points in which the lines diverging from O'' cut the axis of y is $\dfrac{1}{\frac{1}{a} - \frac{1}{b}}$. It is easy also to show, that all the lines measuring from the points of convergence are divided harmonically: the equation to any line, as O_2P_s, is

$$\frac{x}{m} + \frac{cx}{msb} - \frac{y}{sb} = 1,$$

when $$y = 0, \frac{x}{m}\left(1 + \frac{c}{sb}\right) = 1;$$

whence $$\frac{1}{O_1As} = \frac{1}{m - x} = \frac{1}{m} + \frac{sb}{mc}:$$

similarly, $$\frac{1}{O_1Rs} = \frac{1}{m} + \frac{rb}{mc},$$

and so on for the others; and as these are in arithmetic progression, O_1s, O_1r, &c. are in harmonic progression. Instead of taking O_1A as axis, we might have taken any other line, as O_1Q_s, and the result would be the same. And in like manner we might proceed, taking O_2 as the origin, as also O', O'', and, as the cases are similar, we should get the same

result. Hence the property holds for all the lines in the figure. For other problems in Transversals the reader is referred to Carnot's Memoir, or to a small work by Brianchon on the same subject, where he will find some practical applications of the theory.

D. F. G.

ON A NEW FORM OF EQUILIBRIUM OF A REVOLVING FLUID.

In a letter addressed to the French Institute in 1834, M. Jacobi announced the curious discovery, that the form of equilibrium of a fluid mass, consisting of particles attracting each other according to the law of the inverse square of the distance, and revolving about an axis, may be an ellipsoid with three different axes, the shortest being the axis of revo-
[91] lution. A demonstration of this theorem is given by M. Liouville in the *Journal de l' Ecole Polytechnique*, vol. xiv.; but as this work is not easy to be procured, we think a notice of this remarkable discovery will be acceptable to many of our readers. The following demonstration differs considerably from that of Liouville.

Let the fluid mass revolve about the axis of z with an angular velocity ω, and let X, Y, Z be the resolved attractions on a particle at (x, y, z).

The equation to the free surface of the fluid is

$$(X - \omega^2 x)\, dx + (Y - \omega^2 y)\, dy + Z dz = 0.$$

As we know that in the case of an ellipsoid X is of the form Ax, $Y = By$, $Z = Cz$, where A, B, C are independent of the coordinates of the point attracted, we see that this is in the form of the differential equation to an ellipsoid whose axes are a, b, c, or

$$\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2} + \frac{z\,dz}{c^2} = 0,$$

provided $\quad A - \omega^2 = \dfrac{\lambda}{a^2}, \quad B - \omega^2 = \dfrac{\lambda}{b^2}, \quad C = \dfrac{\lambda}{c^2}.$

It remains to find whether these relations will give real values for the axes and angular velocity when a and b are different.

Now $\quad A = \dfrac{3M}{a} \displaystyle\int_0^1 \frac{s^2 ds}{\sqrt{[\{a^2 + (b^2 - a^2)\, s^2\}\, \{a^2 + (c^2 - a^2)\, s^2\}]}}.$

Transform this definite integral into one whose limits are 0 and $\frac{1}{0}$, by putting $s = \frac{a}{\sqrt{(a^2+u)}}$, and we get

$$A = \tfrac{3}{2} M \int_0^\infty \frac{du}{(a^2+u)\sqrt{\{(a^2+u)(b^2+u)(c^2+u)\}}};$$

calling the quantity under the radical D,

$$A = \tfrac{3}{2} M \int_0^\infty \frac{du}{(a^2+u) D},$$

similarly $B = \tfrac{3}{2} M \int_0^\infty \frac{du}{(b^2+u) D}$, $C = \tfrac{3}{2} M \int_0^\infty \frac{du}{(c^2+u) D}$.

Eliminating ω^2 and λ, we get the relation

$$a^2b^2 (B - A) - (a^2 - b^2) c^2 C = 0,$$

or $$(a^2 - b^2) \int_0 \frac{du}{D} \left\{ \frac{1}{\left(1 + \frac{u}{a^2}\right)\left(1 + \frac{u}{b^2}\right)} - \frac{1}{1 + \frac{u}{c^2}} \right\} = 0.$$

If a be different from b, the relation between the axes [92] must satisfy the equation

$$\int_0^\infty \frac{u\,du}{D^3} \left(\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} + \frac{u}{a^2b^2} \right) = 0.$$

Supposing a and b given, this may be considered as an equation to determine c; and as it is negative when $c = 0$, and positive when $c = \frac{1}{0}$, there will be at least one real root of c.

As the above integral is zero, and $\frac{u}{D^3}$ is always positive, and

$$\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} + \frac{u}{a^2b^2}$$

is positive if u be large enough; when u is small, this factor must be negative, or

$$\frac{1}{c^2} > \frac{1}{a^2} + \frac{1}{b^2},$$

and therefore c^2 is less than a^2 or b^2.

To determine the angular velocity, we have

$$(a^2 - b^2)\,\omega^2 = Aa^2 - Bb^2$$

$$= \tfrac{3}{2}\,M\,(a^2 - b^2)\int_0^\infty \frac{u\,du}{(a^2+u)(b^2+u)\,D};$$

if a be different from b,

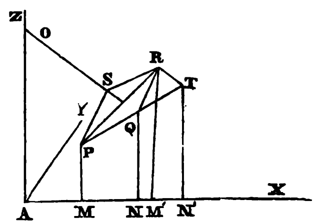
$$\omega^2 = \tfrac{3}{2}\,M\int_0^\infty \frac{u\,du}{(a^2+u)(b^2+u)\,D},$$

a positive quantity. Hence ω is a possible quantity, and there being at least one possible value of c, the problem is possible.

A. S.

RADIUS OF ABSOLUTE CURVATURE.

THE following is a simple method of finding the expression for the radius of absolute curvature, taking the arc as the independent variable. Let PQ, QR be two consecutive elements of the curve, and let us suppose them to be equal; this is the same thing as making ds constant, or s the independent variable. If we complete the parallelogram $PQRS$, the centre of curvature will lie in QS produced, which bisects the angle PQR; and if O be the



[93] centre of curvature it will be the centre of a circle passing through P, Q, R, so that putting $QO = \rho$ and $QS = \lambda$,

$$QR^2 = ds^2 = \lambda\rho.$$

Let α, β, γ be the angles QO makes with the axes, then projecting QS on the axis of x, we have

$$QS\cos\alpha = RT\cos\alpha = M'N'.$$

Now $\quad M'N' = NN' - NM' = MN - NM' = d^2x.$

therefore $\quad \lambda\cos\alpha = d^2x;$

whence $\quad \cos\alpha = \rho\,\dfrac{d^2x}{ds^2}.$

Similarly, $\quad \cos\beta = \rho\,\dfrac{d^2y}{ds^2}, \quad \cos\gamma = \rho\,\dfrac{d^2z}{ds^2}.$

Then from the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, we find

$$\frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2} + \frac{d^2z}{ds^2}\right)},$$

which is the well-known expression.

From these values of $\cos \alpha$, $\cos \beta$, $\cos \gamma$, Cauchy's demonstration of Meusnier's Theorem follows very readily.

Let $u = 0$ be the equation to a surface. Differentiating twice with regard to x, y, z, we have

$$0 = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy + \frac{du}{dz}\,dz$$

$$0 = \frac{du}{dx}\,d^2x + \frac{du}{dy}\,d^2y + \frac{du}{dz}\,d^2z + \&c.$$

The terms we have left out depend only on the values of dx, dy, dz, and therefore only on the *direction* in which these variations are taken, and will be constant if we suppose them taken along a section of the surface made by a plane whose intersection with the tangent plane is constant. If ρ be the radius of curvature of the section, α, β, γ, its inclinations to the axes, we have

$$d^2x = \frac{ds^2}{\rho}\,.\cos \alpha, \&c.$$

Substituting these values

$$0 = \frac{1}{\rho}\,.\left(\cos \alpha\,\frac{du}{dx} + \cos \beta\,\frac{du}{dy} + \cos \gamma\,\frac{du}{dz}\right) + \text{const.}$$

the cosines of the angles which a normal to the surface makes with the axes are proportional to $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$. Hence, calling θ the angle between the normal and the radius of curvature, $\frac{\cos \theta}{\rho} = \text{const.}$ When $\theta = 0$, ρ attains its maximum value $= R$.

Hence $$\rho = R \cos \theta, \qquad [94]$$

or the radius of curvature of an oblique section is the projection of the radius of curvature of the normal section which has the same intersection with the tangent plane.

A. S.

MATHEMATICAL NOTES.

1. SINCE the publication of our first Number, we have met with the following proof, by Professor Jacobi of Koenigsberg, of the equation (K) page 16, which is better than the one there given.

As there, we have

$$\Gamma(m) \,.\, \Gamma(n) = \int_0^\infty \int_0^\infty \varepsilon^{-x-y}\, x^{m-1}\, y^{n-1}\, dx\, dy.$$

Assume $\qquad x + y = r, \ x = rz,$

then, while x and y vary from 0 to ∞, r will vary from 0 to ∞; and since $z = \dfrac{x}{x+y}$, z will vary from 0 to 1. Suppose that x and z, y and r, vary together, then

$$dx = rdz, \quad dy = dr.$$

Substituting these values in the double integral, it becomes

$$\int_0^\infty \varepsilon^{-r}\, r^{m+n-1}\, dr \int_0^1 (1 - z)^{n-1}\, z^{m-1}\, dz,$$

$$= \Gamma\,(m + n) \,.\, \left(\frac{m}{n}\right),$$

whence $$\frac{\Gamma(m) \,.\, \Gamma(n)}{\Gamma\,(m + n)} = \left(\frac{m}{n}\right).$$

γ.

2. *Diagonal of a parallelopiped.*—The following is an easy method of obtaining the expression for the diagonal of a parallelopiped in terms of the edges, and the angles they make with each other.

Let a, b, c be the edges, r the diagonal.

α, β, γ the angles the edges make with each other.

λ, μ, ν the angles the diagonal makes with the edges.

Then projecting the three edges on the diagonal, we have

$$r = a \cos \lambda + b \cos \mu + c \cos \nu.$$

Again, if we project r on a, and compare it with the projection of the broken line a, b, c, which joins its extreme points, on the same line we have the relation

$$r \cos \lambda = a + b \cos \gamma + c \cos \beta.$$

[95] Similarly, $\qquad r \cos \mu = a \cos \gamma + b + c \cos \alpha,$

and $\qquad r \cos \nu = a \cos \beta + b \cos \alpha + c.$

Multiply by a, b, c, and add. Then, by the first condition,

$$r^2 = a^2 + b^2 + c^2 + 2ab \cos \gamma + 2ac \cos \beta + 2bc \cos \alpha.$$

σ.

3. *In the Ellipse* $a'^2 + b'^2 = a^2 + b^2$.—This well-known proposition may be very conveniently proved in the following manner. The equation to a diameter conjugate to that passing through (x', y') is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

Transposing and squaring,

$$\frac{x^2x'^2}{a^4} = \frac{y^2y'^2}{b^4}.$$

Also, $y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$, and at the point where the diameter meets the curve $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$. Substituting these values, the equation becomes

$$\frac{x^2x'^2}{a^4} = 1 - \frac{(x^2 + x'^2)}{a^2} + \frac{x^2x'^2}{a^4},$$

or $$a^2 = x^2 + x'^2.$$

Similarly, $$b^2 = y^2 + y'^2;$$

therefore $$a^2 + b^2 = x^2 + y^2 + x'^2 + y'^2 = a'^2 + b'^2.$$

a.

4. *Problem in the Papers for* 1838.—A quadrant of an ellipse is inscribed in a right-angled triangle having its axes coincident with the sides containing the right angle. Find the curve traced out by its centre of curvature at the point of contact with the hypothenuse, while the point of contact moves from one acute angle to the other.

Let the origin be at the right angle, and let the sides be m, n, and the axes a, b; xy the coordinates of the point of contact, α, β of the centre of curvature. The equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since the hypothenuse is a tangent

$$m = \frac{a^2}{x}, \quad n = \frac{b^2}{y},$$

whence $$\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

Also, we have [96]

$$\frac{x^3}{a^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{y^3}{b^3} = -\frac{b\beta}{a^2 - b^2},$$

therefore $$\frac{a^3}{m^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{b^3}{n^3} = -\frac{b\beta}{a^2 - b^2},$$

whence $$\frac{a^2}{m^2}(a^2 - b^2) = m\alpha, \quad \frac{b^2}{n^2}(a^2 - b^2) = -n\beta.$$

Adding these,

$$\left(\frac{a^2}{m^2} + \frac{b^2}{n^2}\right)(a^2 - b^2) = a^2 - b^2 = m\alpha - n\beta.$$

Again, multiplying the first by m^2 and the second by n^2, and subtracting,

$$(a^2 - b^2)^2 = m^3\alpha + n^3\beta,$$

whence $$m^3\alpha + n^3\beta = (m\alpha - n\beta)^2,$$

the required locus, which is a parabola. π.

5. The area of the parallelogram formed by tangents applied at the extremities of any two conjugate diameters of an ellipse is constant.

Let x, y be the coordinates of the extremity of one diameter, and x', y' those of the other; θ, θ' the angles which they make with x. Then

$$x = a' \cos\theta, \quad y = a' \sin\theta, \quad x' = b' \cos\theta', \quad y' = b' \sin\theta'.$$

The tangent whose inclination to the axis of x is θ, passes through the point $x'y'$; therefore, by the equation in page 9, substituting $\tan\theta$ for α, and multiplying by $\cos\theta$,

$$y' \cos\theta - x' \sin\theta = \sqrt{\{(b\cos\theta)^2 + (a\sin\theta)^2\}}.$$

Multiply by a', and substitute for x' and y' their values in b' and θ', therefore

$$a'b' \sin(\theta' - \theta) = \sqrt{\{(b^2(a'\cos\theta)^2 + a^2(a'\sin\theta)^2\}}$$
$$= \sqrt{(b^2x^2 + a^2y^2)} = ab.$$

γ.

[97]

ON THE PROPAGATION OF A WAVE IN AN ELASTIC MEDIUM.

THE following investigation of the equation for the propagation of a wave in an elastic medium, may be considered as supplemental to the two articles on the Undulatory Theory in the first and second Numbers of this Journal: though perhaps it ought more properly to have preceded them. But as we do not pretend to offer a complete treatise on the subject, the order of the articles is not a matter of much consequence.

1. Light, according to the wave theory, consists of undulations propagated in an elastic medium. From the laws of the interference of polarized light, it appears that the constituent vibrations are transverse to the direction of propagation; that polarized light consists of vibrations in one direction, or perpendicular to one plane, which in Fresnel's theory is the *plane of polarization*, and in Cauchy's theory is perpendicular to the *plane of polarization;* and thirdly, that there are no vibrations, or at least none capable of producing the sensation of light in the direction of propagation. Fresnel assigns as the reason for this, that the resistance to compression of the medium is so great, that if a disturbance takes place motion is propagated to a great distance before the disturbed molecule returns to rest. Hence, a wave of great breadth will be propagated, but of feeble intensity, as the *vis viva* remains constant.

2. To this we may add, that the possibility of the propagation of motion to a distance depends upon this, that waves of *constant breadth* can be propagated: if we examine our equations, we see that this depends upon the disturbance being a function of $vt - x$; and the partial differential [98] equation which gives this form of solution is obtained, in the simplest case, namely that of motion in one direction by supposing the elastic force proportional to the condensation, and the condensations to be infinitesimal. Without these restrictions the equations cannot be solved; but from several considerations it seems probable, that a particle once disturbed does not immediately come to rest, and that the waves increase in breadth. We may see examples of this in the waves propagated when a stone is thrown into still water. But in water the velocity of a wave depends on its breadth; and thus, though the breadths of the waves increase, they do not interfere with each other.

3. It appears probable, from the change in the character of sounds caused by distance, the loss of sharpness and distinctness, that the breadth of a wave of air increases. If we suppose the same to be the case with those waves of light which consist of vibrations in the direction of propagation, but to a greater degree than in sound, we shall have, in the first place, a rapid diminution in the *vis viva* of each wave: and also, as their velocities are nearly independent of their breadths, each wave will overtake its predecessor and be overtaken by its successor, and there will be an interference which may speedily obliterate all traces of the disturbance.

4. We have now to consider that complex system of disturbances which constitutes an undulation.

We shall suppose the molecules of the media which we consider to be so arranged, that the three rectangular directions, or the three axes of elasticity, have the same direction at every point. These we shall take for coordinate axes, and shall suppose that the elasticity called into play is greatest for a displacement parallel to the axis of x, and least for one parallel to the axis of z, or that the axes of x, y, z are the axes of greatest, mean, and least elasticity.

5. If we suppose a range of particles, in a plane parallel to that of yz, to be moved upwards, or parallel to the axis of z, the resultant of the attraction of all the others will tend to pull it back along the same direction with a force proportional to the displacement: the same thing will happen if it receive a displacement parallel to the axis of y, but for the same displacement the force of restitution will be greater than before, as we have supposed the elasticity for displacements parallel to y to be greater than for those parallel to z.

Let a^2, b^2, c^2 be the coefficients of elasticity, so that in the first case the force tending to bring back the range of particles is $c^2 \times$ displacement, and in the other $b^2 \times$ displacement. If the influence of one range of particles does not extend beyond the nearest range, the force of each of the adjacent ranges will pull the displaced range back with a force $= \frac{c^2}{2} \times$ displacement.

Let us now consider a succession of ranges to be displaced by different quantities, all parallel to the axis of z, the ranges [99] being all at equal intervals. Let the range at a distance x from the origin be displaced by a quantity z_x, that at a distance $x + h$ by a quantity z_{x+h}, and so on. The range at x will be pulled *up* by the range at $x + h$ with a force

$$\tfrac{1}{2} c^2 (z_{x+h} - z_x) = \tfrac{1}{2} c^2 \Delta z_x ,$$

and *down* by the range at $x - h$ with a force $\frac{1}{2} c^2 \Delta z_{x-h}$: thus from these two it will be pulled *up* with a force equal to their difference, or

$$\tfrac{1}{2} c^2 (\Delta z_x - \Delta z_{x-h}) = \tfrac{1}{2} c^2 \Delta^2 z_{x-h} ;$$

and if M be the mass of the range at x, we have for its motion

$$\frac{d^2 z_x}{dt^2} = \frac{c^2}{2M} \Delta^2 z_{x-h} .$$

6. The general solution of this equation of mixed partial differences has been given in Article II. of our last Number; and there it is shown, that the velocity of propagation depends on the length of the wave, and thus the phenomenon of dispersion is accounted for. In the present case we shall take the approximate solution, where h is very small. By putting $\frac{d^2 z_x}{dx^2} h^2$ for $\Delta^2 z_{x-h}$, our equation becomes

$$\frac{d^2 z_x}{dt^2} = c^2 \frac{h^2}{2M} \frac{d^2 z_x}{dx^2}.$$

This is the well-known equation, which shows that a wave will be propagated along the axis of x with a velocity $c \frac{h}{\sqrt{(2M)}}$. If the vibrations take place parallel to the axis of y, there will be a wave propagated with a velocity $b \frac{h}{\sqrt{(2M)}}$. Fresnel states that he has satisfied himself, by experiment, that the velocity of a plane wave depends solely on the direction of the constituent vibrations, and not on the direction of the wave. It would appear from this, that the factors of b and c in the above expressions must be constant for different positions of the wave; and it results, that plane waves, consisting of vibrations parallel to the axes of x, y, or z, will be propagated perpendicularly to themselves with velocities which are proportional to a, b, c, or to the square roots of the coefficients of the elastic force for the disturbance of a single range.

7. If the range of particles in the plane of yz receive a disturbance in that plane, but not parallel to one of the axes, we must resolve this displacement into *two*, one parallel to each axis, and there will result *two* waves, propagated with velocities b and c respectively along the axis of x, and corresponding waves propagated in a negative direction.

And this result follows, whether the displacement takes place along a line, as in polarized light, or in some more complicated manner, as in common light; and thus any kind of disturbance in one of the coordinate planes will produce two waves, polarized at right angles to each other, and of different refrangibility.

8. We have now to consider the general case of [100] a plane wave inclined at any angles to the coordinate planes. A vibration in this plane will produce a force of restitution, which will have a different direction from the displacement. Resolve this force into two, one along the direction of dis-

placement, the other perpendicular to it: if this last portion be also perpendicular to the front of the wave, it may be neglected, and a wave will be propagated with a velocity proportional to the square root of the elasticity resolved along the direction of vibration; if it be not perpendicular to the front of the wave, it cannot be neglected. We have, therefore, first to inquire whether this is the case for any direction of vibration in the plane; and this leads us into the subject which was discussed in our last Number.

A. S.

CIRCULAR SECTIONS IN SURFACES OF THE SECOND ORDER.

In determining the circular sections in the Surface of Elasticity, Fresnel has made use of a method which is very readily applicable to surfaces of the second order; and as it has not yet been introduced into any work on Analytical Geometry, it may be useful to insert it here.

Taking first the surfaces which have a centre, let their equation be

$$Px^2 + P'y^2 + P''z^2 = H \dots\dots\dots\dots (1),$$

and let this be cut by a plane

$$z = mx + ny \dots\dots\dots\dots (2),$$

which we suppose to pass through the centre, as all sections made by parallel planes are similar. Let this plane also cut the sphere

$$x^2 + y^2 + z^2 = r^2 \dots\dots\dots\dots (3).$$

Now, as m, n, r are indeterminate, we can so assume the position of the plane and the magnitude of the sphere, that the circular section of the surface (1), if it exist, shall coincide with the section of the sphere; and if these coincide, the equations to their projections on the plane of xy must be identical, which gives us conditions for determining m and n. Substituting for z in (1) and (3) its value from (2), we get

$$(P + P''m^2)\, x^2 + (P' + P''n^2)\, y^2 + 2P''mn\, xy = H \dots (4),$$

$$(1 + m^2)\, x^2 + (1 + n^2)\, y^2 + 2mn\, xy = r^2 \dots (5).$$

Comparing each term separately, those involving xy will coincide if either

$$m = 0, \quad n = 0, \quad \text{or} \quad r^2 = \frac{H}{P''}.$$

Taking the first condition and comparing the other terms, [101]

$$\frac{H}{P} = r^2, \quad \text{and} \quad \frac{P' + P''n^2}{H} = \frac{1 + n^2}{r^2},$$

which gives $$P' + P''n^2 = P(1 + n^2),$$

and $$n = \pm \sqrt{\left(\frac{P - P'}{P'' - P}\right)}.$$

If we suppose $n = 0$, we find in the same manner

$$m = \pm \sqrt{\left(\frac{P' - P}{P'' - P'}\right)}.$$

The third condition leads to no result, and therefore is not to be considered.

In the ellipsoid, P, P', P'' are all positive, and

$$P < P' < P''.$$

This shows that the value of n is impossible, and that of m possible; therefore there are two directions arising from the doubtful sign in which the ellipsoid may be cut in circular sections, determined by the equation to the cutting plane,

$$z = \pm \sqrt{\left(\frac{P' - P}{P'' - P'}\right)} x.$$

In the hyperboloid of one sheet P'' is negative, and the value of n is possible and m impossible. In the hyperboloid of two sheets P' and P'' are both negative, and $P'' < P'$, m is possible and n impossible. It is true, that for a plane passing through the centre the section is impossible, but a plane drawn parallel to this at a sufficient distance from the centre will cut the surface in a circle.

The equation to the surfaces without a centre is

$$p'y^2 + pz^2 = pp'x.$$

Let this be cut by a plane

$$x = mz + ny,$$

which also cuts the sphere

$$x^2 + y^2 + z^2 = 2rx.$$

The equations to the projections of the sections on the plane of zy are

$$p'y^2 + pz^2 - mpp'z - npp'y = 0,$$

$$(1 + n^2)\, y^2 + (1 + m^2)\, z^2 + 2mnzy - 2mrz - 2nry = 0.$$

In order that these may coincide, the term involving zy must vanish, which will be the case if $m = 0$ or $n = 0$.

If $m = 0$, then $1 + n^2 = \frac{p'}{p}$ and $n = \pm\sqrt{\left(\frac{p'-p}{p}\right)}$.

If $n = 0$, then $1 + m^2 = \frac{p}{p'}$ and $m = \pm\sqrt{\left(\frac{p-p'}{p'}\right)}$.

[102] In the elliptic paraboloid p and p' are both positive, and according as p' is greater or less than p, the first or second is to be taken, the other becoming impossible. In either case there are two series of circular sections corresponding to the positive and negative sign.

In the hyperbolic paraboloid p or p' is negative, so that there are no sections in which it is cut in a circle. This would appear also from the nature of the surface, as it can never be cut by a plane in a closed curve.

The same method may be applied to the oblique cone, so as to determine the sub-contrary sections.

By determining the circular sections in this manner, it is seen at once that any two belonging to different series are situate on the same sphere.

D. F. G.

PROPOSITIONS IN THE THEORY OF NUMBERS.

The proposition, "The continued product of any m consecutive integers is divisible by $1.2.3 \ldots m$," has been strictly proved by induction; but this mode of proof only shows, if we may so speak, that the proposition *must* be true, without showing *why* it is true.

We have seen it reasoned, that since the product of any m consecutive integers is divisible by each of the natural numbers up to m separately, it is divisible by their product: but a little consideration will show that this does not follow.

The problem from which the above proposition is here deduced, has been proposed and solved before, but not, that we are aware, in the same form.

Prop. 1. If p be a prime number, the index of the highest power of p that is a divisor of the continued product $1.2.3 \ldots M$, is $\frac{M - S(M)}{p-1}$, where $S(M)$ represents the sum of the digits of M expressed in the scale whose radix is p.

We shall adopt the notation used by several Continental mathematicians, $M!$ for $1.2.3 \dots M$.

Let the quotient of M by p be M_1, and the remainder a_0, then

$$M = M_1 p + a_0,$$

and the multiples of p in the series 1, 2, 3, &c. M, are

$$p,\ 2p,\ 3p,\ \text{\&c. } M_1 p:$$

wherefore, if K denote the product of the remaining numbers in the former series,

$$\begin{aligned} M! &= K \times p.2p.3p \dots M_1 p, \\ &= Kp^{M_1}.M_1! \end{aligned}$$

Let r be the index of the greatest power of p in $M!$ [103]

r_1 . . . . . . $M_1!$,

then from the last equation

$$r = M_1 + r_1.$$

For the same reason, if

$$M_1 = M_2 p + a_1, \quad \text{and} \quad M_2 = M_3 p + a_2, \quad \text{\&c.},$$

and r_2, r_3, &c. be corresponding quantities for $M_2!$, $M_3!$, &c.

$$\begin{aligned} r_1 &= M_2 + r_2, \\ r_2 &= M_3 + r_3, \\ &\dots \\ r_{m-1} &= M_m + r_m, \\ r_m &= 0. \end{aligned}$$

M_m being that quotient less than p at which we must finally arrive. Adding all these equations,

$$r = M_1 + M_2 + M_3 + \dots + M_m.$$

But, adding the equations

$$\begin{aligned} M &= M_1 p + a_0, \\ M_1 &= M_2 p + a_1, \\ M_2 &= M_3 p + a_2, \\ &\dots \\ M_{m-1} &= M_m p + a_{m-1}, \\ M_m &= a_m; \end{aligned}$$

and observing that a_0, a_1, &c. a_m are the successive digits of M expressed in the scale of p, we have

$$M + M_1 + M_2 + \dots + M_m = (M_1 + M_2 + \dots + M_m) p + S(M);$$

whence $$M_1 + M_2 + \dots + M_m = \frac{M - S(M)}{p - 1},$$

and the proposition is proved.

The following proposition is also useful.

Prop. 2. If M and N be two numbers, of which M is the greater, and $S(M)$, $S(N)$ the sums of their digits to any radix r, $S(M-N)$ is not less than $S(M) - S(N)$.

For let $M = a_m r^m + a_{m-1} r^{m-1} + \ldots + a_n r^n + a_{n-1} r^{n-1} + \ldots,$
and $\quad N = \qquad\qquad\qquad\qquad b_n r^n + b_{n-1} r^{n-1} + \ldots,$
then $M - N = a_m r^m + a_{m-1} r^{m-1} + \ldots + (a_n - b_n) r^n + (a_{n-1} - b_{n-1}) r^{n-1} + \ldots$

Now, as long as the digits of N are not greater than the corresponding digits of M, the digits of $M-N$ will be the excesses of those of M over those of N. But suppose b_x greater than a_x, then we form the digits of $M-N$ by writing that part of the difference thus,

$$(a_{x+1} - b_{x+1} - 1) r^{x+1} + (r + a_x - b_x) r^x,$$

[104] and the sum of these two digits is

$$a_{x+1} - b_{x+1} + a_x - b_x + r - 1;$$

hence $S(M-N)$ may be greater, but cannot be less, than

$$S(M) - S(N).$$

Prop. 3. The product of any m consecutive integers is divisible by $1.2.3 \ldots m$.

By Prop. 1, the index of p in $N!$ is $\dfrac{N - S(N)}{p-1}$, and in $(N+m)!$, $\dfrac{N + m - S(N+m)}{p-1}$; hence the index of p in

$$(N+1)(N+2)\ldots(N+m) \quad \text{or} \quad \frac{(N+m)!}{N!},$$

is the difference of these quantities, or

$$\frac{m - S(N+m) + S(N)}{p-1}.$$

The index of p in $m!$ is $\dfrac{m - S(m)}{p}$; and, by Prop. 2, making

$$M = N + m, \quad S(m) \text{ is not } < S(N+m) - S(N),$$

therefore $\; m - S(m)$ is not $> m - S(N+m) + S(N)$;

and the index of p in $m!$ is not greater than the index of p in $(N+1)(N+2)\ldots(N+m)$. Hence, if

$$(N+1)(N+2)\ldots(N+m) = 2^a.3^b.5^c\ldots,$$

and $\qquad 1.2\,3\ldots m = 2^\alpha.3^\beta.5^\gamma\ldots,$

α is not $> a$, β not $> b$, γ not $> c$, &c. Consequently

$(N+1)(N+2)\ldots(N+m)$ is divisible by $1.2.3\ldots m$.

S. S. G.

NOTES ON FOURIER'S HEAT.

THE method employed by Fourier to integrate the partial differential equations which occur in the Theory of Heat, is to assume some simple form of a singular solution, and afterwards to extend it so as to include all the circumstances of the problem. It is in effecting this that he has displayed the great resources of his analysis, and imparted so great an interest to his work by the variety and ingenuity of his methods. Indeed there is a freshness and originality in the writings of Fourier which make them in no ordinary degree arrest the attention of the reader. But however much we may admire the means by which Fourier has overcome the difficulties of the problems he had to deal with, yet it seems more agreeable to the usual style of mathematical [105] investigation to deduce a result by limiting the general solution by means of the conditions of the problem, than by extending a particular case.

That this may be sometimes done with even more readiness than by Fourier's method, will be seen by the following solution of a problem given in p. 161 of the *Théorie de la Chaleur*. We may remark, that there is in general no difficulty in the solution of the partial differential equations, but only in the proper determination of the arbitrary functions in the solution, so as to suit the conditions of the problem.

If a rectangular plate, bounded by two infinite parallel edges, have one of its extremities kept at a constant temperature 1, while the infinite edges perpendicular to the heated edge are retained at a constant temperature 0, the equation from which the temperature is to be determined is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1),$$

where v is the temperature at the point x, y, the origin being at the middle point of the heated edge, the axis of x bisecting the plate, and the axis of y parallel to the heated edge. For the sake of shortness Fourier represents the breadth of the plate by π.

The solution of the equation (1) by the method of the separation of the symbols of operation from those of quantity, is

$$v = \cos\left(y\,\frac{d}{dx}\right)\phi(x) + \sin\left(y\,\frac{d}{dx}\right)\psi(x)\ldots \quad (2),$$

$\phi(x)$ and $\psi(x)$ being arbitrary functions of x. And it may also be put under the form

$$v = F\{x + y\sqrt{(-1)}\} + f\{x - y\sqrt{(-1)}\},$$

where $F(x) = \frac{1}{2}\{\phi(x) + \psi(x)\}$ and $f(x) = \frac{1}{2}\{\phi(x) - \psi(x)\}$.

Now, on looking at the circumstances of the problem, it will be seen that it must be subject to the following conditions:

1st. v must be symmetrical with regard to y and $-y$.

2nd. $v = 0$ when $y = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, whatever x may be.

3rd. $v = 1$ when $x = 0$, whatever y may be.

4th. v must be very small when x is very large.

From the first condition we must have $\psi(x) = 0$, as otherwise the second term would change its sign when $-y$ is put for y. Hence we have only

$$v = \cos\left(y\frac{d}{dx}\right)\phi(x) \ldots\ldots\ldots\ldots \quad (3).$$

By the second condition, putting $\frac{\pi}{2}$ for y in equation (3), we have

$$0 = \cos\left(\frac{\pi}{2}\frac{d}{dx}\right)\phi(x) \ldots\ldots\ldots\ldots (4).$$

[106] Now this is in fact a linear differential equation with constant coefficients, and of an infinite order. By the principles laid down in Art. v. of our first Number, we can integrate this equation if we know the roots of the equation $\cos\left(\frac{\pi}{2}z\right) = 0$. Now these are

$$\pm 1, \pm 3, \pm 5, \text{\&c.},$$

being in number infinite. Hence the solution of (4) is

$$\phi(x) = \left\{\begin{array}{l} C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \text{\&c.} \\ + C'_1\varepsilon^{x} + C'_3{}^{3x}\varepsilon + C'_5\varepsilon^{5x} + \text{\&c.}\end{array}\right\} \ldots\ldots (5),$$

the number of terms and arbitrary constants being infinite. By the fourth condition it appears that the second line of (5) must disappear, as otherwise v would be very large when x is very large. Hence we must have

$$C'_1 = 0, \quad C'_3 = 0, \quad C'_5 = 0, \text{ \&c.};$$

and equation (5) is reduced to

$$\phi(x) = C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \text{\&c.} \ldots\ldots (6),$$

and v becomes

$$v = \cos\left(y\frac{d}{dx}\right)(C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \&c.) \ldots (7).$$

By the third condition $v=1$ when $x=0$. If then we expand the symbol of operation in (7), operate on each term separately, and then make $x=0$, we shall find

$$1 = C_1 \cos y + C_3 \cos 3y + C_5 \cos 5y + \&c. \ldots .(8),$$

where y is contained between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

In order to determine the arbitrary constants, we shall follow Fourier's method of definite integrals. If we multiply both sides of (8) by $\cos y\, dy$, and integrate between the limits $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, all the terms except the first will disappear, as they can each be decomposed into the cosines of even multiples of y, which, on integration, vanish at both limits. Hence we have

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy \cos y = C_1 \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy \cos^2 y = \frac{C_1}{2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy\,(1 + \cos 2y),$$

whence we find $C_1 = \frac{4}{\pi}$.

In a similar manner we should find

$$C_3 = -\frac{1}{3}\,\frac{4}{\pi}, \quad C_5 = -\frac{1}{5}\,\frac{4}{\pi}, \quad C_7 = -\frac{1}{7}\,\frac{4}{\pi},$$

and so on.

Substituting these values in equation (7), it becomes [107]

$$\frac{\pi}{4}\, v = \cos\left(y\frac{d}{dx}\right)\left(\varepsilon^{-x} - \frac{1}{3}\,\varepsilon^{-3x} + \frac{1}{5}\,\varepsilon^{-5x} - \frac{1}{7}\,\varepsilon^{-7x} + \&c.\right)$$

Now if we expand the sign of operation, and apply it to such a term as ε^{-nx}, we shall find that it becomes $\cos ny\, \varepsilon^{-nx}$. Hence the expression for v becomes

$$\frac{\pi}{4}\, v = \varepsilon^{-x} \cos y - \tfrac{1}{3}\,\varepsilon^{-3x} \cos 3y + \tfrac{1}{5}\,\varepsilon^{-5x} \cos 5y - \&c.$$

which is one form of the solution which Fourier gives. It may easily be reduced to a more simple form. For if we substitute for the cosines their exponential values, we have

$$\frac{\pi}{2}\, v = \begin{cases} \varepsilon^{-[x-y\sqrt{(-1)}]} - \frac{1}{3}\,\varepsilon^{-3\,[x-y\sqrt{(-1)}]} + \frac{1}{5}\,\varepsilon^{-5\,[x-y\sqrt{(-1)}]} - \&c. \\ + \,\varepsilon^{-[x+y\sqrt{(-1)}]} - \frac{1}{3}\,\varepsilon^{-3\,[x+y\sqrt{(-1)}]} + \frac{1}{5}\,\varepsilon^{-5\,[x+y\sqrt{(-1)}]} - \&c. \end{cases}$$

which, by Gregorie's series, become

$$\frac{\pi}{2} v = \tan^{-1} \varepsilon^{-[x-y\sqrt{(-1)}]} + \tan^{-1} \varepsilon^{-[x+y\sqrt{(-1)}]}$$

$$= \tan^{-1} \frac{\varepsilon^{-[x-y\sqrt{(-1)}]} + \varepsilon^{-[x+y\sqrt{(-1)}]}}{1 - \varepsilon^{-2x}}$$

$$= \tan^{-1} \left(\frac{2 \cos y}{\varepsilon^{x} - \varepsilon^{-x}} \right),$$

which is the simplest form that the expression can assume.

D. F. G.

DEMONSTRATIONS OF SOME PROPERTIES OF SURFACES OF THE SECOND ORDER.

To the Editor of the Cambridge Mathematical Journal.

SIR,—I observe the author of a Paper in your first Number says, he is not aware that any person has made use of the symmetrical form of the equations to a straight line: he will find them employed in Cauchy's "*Leçons sur les Applications du Calcul Infinitésimal.*"

Perhaps the following solutions of two common problems may be acceptable to some of your readers.

1. To find the locus of the bisections of parallel chords in the surface whose equation is

$$Ax^2 + By^2 + Cz^2 = k.$$

[108] Suppose any one of the chords meets the surface in the points whose coordinates are $x_0, y_0, z_0, x_1, y_1, z_1$; then if α, β, γ are the angles which the chord makes with the three axes, we have

$$\frac{x_0 - x_1}{\cos \alpha} = \frac{y_0 - y_1}{\cos \beta} = \frac{z_0 - z_1}{\cos \gamma} \; \dots\dots\dots\dots \; (1).$$

Let x, y, z be the coordinates of the middle point of the chord; then

$$x = \frac{x_0 + x_1}{2}, \quad y = \frac{y_0 + y_1}{2}, \quad z = \frac{z_0 + z_1}{2},$$

or

$$\frac{x_0 + x_1}{x} = \frac{y_0 + y_1}{y} = \frac{z_0 + z_1}{z};$$

therefore combining these equations with (1) we have

$$\frac{x_0^2 - x_1^2}{x \cos \alpha} = \frac{y_0^2 - y_1^2}{y \cos \beta} = \frac{z_0^2 - z_1^2}{z \cos \gamma} \; \dots\dots\dots \; (2).$$

Now, from the equation to the surface, we have

$$A(x_0^2 - x_1^2) + B(y_0^2 - y_1^2) + C(z_0^2 - z_1^2) = 0,$$

which combined with (2), gives immediately

$$Ax \cos \alpha + By \cos \beta + Cz \cos \gamma = 0,$$

which is the equation to the locus required.

2. To find the locus of the intersection of tangent planes drawn parallel to any system of conjugate diameters in the same surface.

I shall first observe, that the projection of the diagonal of a parallelogram on any line whatever, is evidently equal to the sum of the projections of any two contiguous sides on the same line. Whence it easily follows, that the projection of the diagonal of a parallelopiped is equal to the sum of the projections of any three of its edges, of which no two are parallel to each other.

Hence, if ξ, η, ζ are three straight lines drawn from the origin to three points P_0, P_1, P_2, whose coordinates are x_0, y_0, z_0, x_1, y_1, z_1, x_2, y_2, z_2, and if through the points P_0, P_1, P_2, we draw planes parallel respectively to the planes of $\eta\zeta$, $\xi\zeta$, $\xi\eta$, these three planes will intersect in some point P, and a line drawn from the origin to P will be the diagonal of a parallelopiped, of which ξ, η, ζ are three edges. And if we call x, y, z the coordinates of P, and project the diagonal and the three lines ξ, η, ζ on the three axes successively, we have evidently, by the principle above mentioned,

$$\left.\begin{aligned} x &= x_0 + x_1 + x_2 \\ y &= y_0 + y_1 + y_2 \\ z &= z_0 + z_1 + z_2 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (3).$$

Now suppose ξ, η, ζ are three semiconjugate diameters of the surface proposed, we have of course

$$Ax_1x_2 + By_1y_2 + Cz_1z_2 = 0,$$
$$Ax_0x_2 + By_0y_2 + Cz_0z_2 = 0,$$
$$Ax_0x_1 + By_0y_1 + Cz_0z_1 = 0;$$

and if we multiply these equations by 2, and add [109]
them to the three following, viz.

$$Ax_0^2 + By_0^2 + Cz_0^2 = k,$$
$$Ax_1^2 + By_1^2 + Cz_1^2 = k,$$
$$Ax_2^2 + By_2^2 + Cz_2^2 = k,$$

we obtain immediately

$$A(x_0+x_1+x_2)^2+B(y_0+y_1+y_2)^2+C(z_0+z_1+z_2)^2=3k,$$

or by (3), $$Ax^2+By^2+Cz^2=3k,$$

which is evidently the equation to the locus required.

I do not know whether this solution is new, but I do not recollect to have seen it before.

The equations (3) lead immediately to the common formulæ for the transformation of coordinates; for if we call $\alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, the angles which ξ, η, ζ make with the three axes respectively, we have

$$x_0=\xi\cos\alpha_0,\ y_0=\xi\cos\beta_0,\ z_0=\xi\cos\gamma_0,\ \&c.,$$

and the equations (3) become

$$\begin{aligned} x &= \xi\cos\alpha_0+\eta\cos\alpha_1+\zeta\cos\alpha_2,\\ y &= \xi\cos\beta_0+\eta\cos\beta_1+\zeta\cos\beta_2,\\ z &= \xi\cos\gamma_0+\eta\cos\gamma_1+\zeta\cos\gamma_2,\end{aligned}$$

where ξ, η, ζ are evidently the coordinates of P, referred to lines drawn from the origin through P_0, P_1, P_2 as axes.

I am, Sir, &c. M. N. N.

Oxford, Jan. 31, 1838.

ON GENERAL DIFFERENTIATION. NO. II.

In our first Number we investigated the general differential coefficients of ε^{nx} and x^n. We shall now proceed to find those of other simple functions, and to give some examples of the application of the theory.

1. We found that our formulæ gave an infinite value for $\dfrac{d^\alpha x^n}{dx^\alpha}$, when n is not, and $n-\alpha$ is a positive integer. But by using the complementary function, a finite value of another form may be obtained, in the same manner as the value of $\int\dfrac{dx}{x}$ may be derived from that of $\int x^n dx$.

Let $\alpha-n$ in formula (N), and $n-\alpha$ in formula (O), page (19), be assumed equal to $r+\beta$, when r is an integer,
[110] then the second member of each will be a multiple of

$$\frac{x^{r+\beta}}{\sin(r+\beta)\pi},$$

which is infinite when $\beta=0$, but by taking in a part of the

complementary function becomes $\dfrac{x^r(x^\beta - a^\beta)}{\sin(r+\beta)\pi}$, which is a vanishing fraction whose value is $\dfrac{(-1)^r}{\pi}x^r \log\dfrac{x}{a}$ when $\beta = 0$. Hence in these cases (N) and (O) become respectively

$$\frac{d^{-a}}{dx^{-a}}\frac{1}{x^n} = (-1)^{1+n}\frac{1}{\Gamma(1+a-n)\Gamma(n)}x^{a-n}\log\frac{x}{a} \dots. (Q),$$

$$\frac{d^a x^n}{dx^a} = (-1)^n\frac{\sin n\pi}{\pi}.\frac{\Gamma(1+n)}{\Gamma(1+n-a)}x^{n-a}\log\frac{x}{a} \dots. (R).$$

2. The value of $\dfrac{d^a \log x}{dx^a}$ may be readily derived from the formulæ for the differential coefficients of powers of x. For since

$$\frac{d}{dx}\log x = \frac{1}{x},$$

$$\frac{d^a}{dx^a}\log x = \frac{d^{a-1}}{dx^{a-1}}\frac{1}{x}.$$

Hence, if a be positive,

$$\frac{d^a}{dx^a}\log x = (-1)^{a-1}\Gamma(a)\frac{1}{x^a}, \text{ by } (E) \dots\dots\dots (S),$$

and $\dfrac{d^{-a}}{dx^{-a}}\log x = (-1)^a\dfrac{\pi}{\sin a\pi \,.\, \Gamma(1+a)}x^a$, by ($N$)...($T$).

3. If it be known that

$$\frac{d^a}{dx^a}\phi(x) = \psi(x),$$

it is evident that

$$\frac{d^a}{dx^a}\phi(mx+a) = m^a\psi(mx+a);$$

hence the general differential coefficients of all rational functions of x can be found by preparing them as for integration. Also, if the differential coefficient of a function to any integral index be a rational function, its general differential coefficient may be found. As an example, let

$$y = \tan^{-1} x,$$

then $\qquad \dfrac{dy}{dx} = \dfrac{1}{1+x^2} = \frac{1}{2}\left\{\dfrac{1}{1+\sqrt{(-1)}\,x} + \dfrac{1}{1-\sqrt{(-1)}\,x}\right\};$

therefore, using the formula (P),

$$\frac{d^\alpha y}{dx^\alpha} = \tfrac{1}{2}\frac{P(-1)}{P(-\alpha)}\left[\frac{\{\sqrt{(-1)}\}^\alpha}{\{1+\sqrt{(-1)}\,x\}^\alpha} + \frac{\{-\sqrt{(-1)}\}^\alpha}{\{1-\sqrt{(-1)}\,x\}^\alpha}\right].$$

[111] Substituting for x its value $\tan y$, observing that

$$\sqrt{(-1)} = \cos(2r+\tfrac{1}{2})\,\pi + \sqrt{(-1)}\sin(2r+\tfrac{1}{2})\,\pi,$$

and reducing by De Moivre's theorem, we obtain

$$\frac{d^\alpha y}{dx^\alpha} = \frac{P(-1)}{P(-\alpha)}(\cos y)^\alpha \cos\alpha\,\{y-(2r+\tfrac{1}{2})\,\pi\}.$$

The value of $\dfrac{P(-1)}{P(-\alpha)}$ is that of $x^\alpha.\dfrac{d^{\alpha-1}x^{-1}}{dx^{\alpha-1}}$, which is

$$(-1)^{\alpha-1}\,\Gamma(\alpha)$$

when α is positive, and

$$(-1)^\alpha\,\frac{\pi}{\sin(-\alpha\pi).\Gamma(1-\alpha)}$$

when α is negative. The formula fails when α is a negative integer.

4. Let us ascertain in what cases the differential coefficient of a constant quantity is not zero.

Since $C = Cx^0$,

$$\frac{d^\alpha C}{dx^\alpha} = C\,\frac{P(0)}{P(-\alpha)}\,x^{-\alpha} = \frac{C}{P(-\alpha)}\,x^{-\alpha},$$

which will not be zero if $P(-\alpha)$ be not infinite, which is only when $-\alpha$ is a positive integer, or α a negative integer, that is, in the case of common integration.

5. The differential coefficient of infinity may be finite. For $\infty = CP(m)\,x^n$, if m be not a positive integer: therefore

$$\frac{d^\alpha\infty}{dx^\alpha} = C\,\frac{P(m)\,P(n)}{P(n-\alpha)}\,x^{n-\alpha},$$

which may be finite if n be a positive integer, and $n-\alpha$ not.

6. We proceed to find the general differential coefficients of $\cos x$ and $\sin x$.

Since $\cos x + \sqrt{(-1)}\sin x = \epsilon^{\sqrt{(-1)}x}$

$$\frac{d^\alpha \cos x}{dx^\alpha} + \sqrt{(-1)}\,\frac{d^\alpha \sin x}{dx^\alpha} = \{\sqrt{(-1)}\}^\alpha\,\epsilon^{\sqrt{(-1)}x}$$

$$= \{\cos(2r+\tfrac{1}{2})\,\pi + \sqrt{(-1)}\sin(2r+\tfrac{1}{2})\,\pi\}^\alpha\,\{\cos x + \sqrt{(-1)}\sin x\}$$

$$= \cos\{(2r+\tfrac{1}{2})\,\alpha\pi + x\} + \sqrt{(-1)}\sin\{(2r+\tfrac{1}{2})\,\alpha\pi + x\}.$$

In like manner,

$$\frac{d^a \cos x}{dx^a} - \sqrt{(-1)}\,\frac{d^a \sin x}{dx^a}$$
$$= \cos\left\{(2r' + \tfrac{1}{2})\,a\pi + x\right\} - \sqrt{(-1)}\sin\left\{(2r' + \tfrac{1}{2})\,a\pi + x\right\},$$

r' being some integer, not necessarily the same as r. Adding and subtracting,

$$\left.\begin{aligned}
&\frac{d^a \cos x}{dx^a}\\
&= \left\{\cos(r - r')\,a\pi + \sqrt{(-1)}\sin(r - r')\,a\pi\right\}\\
&\qquad\qquad \cos\left\{(r + r' + \tfrac{1}{2})\,a\pi + x\right\},\\
&\frac{d^a \sin x}{dx^a}\\
&= \left\{\cos(r - r')\,a\pi + \sqrt{(-1)}\sin(r - r')\,a\pi\right\}\\
&\qquad\qquad \sin\left\{(r + r' + \tfrac{1}{2})\,a\pi + x\right\},
\end{aligned}\right\}\ldots.(U). \qquad [112]$$

Since r and r' are both arbitrary, there is no relation between $r + r'$ and $r - r'$, except that they must be both odd or both even. If a be a rational fraction whose denominator, when in its lowest terms, is n, $r + r'$ must go through all the $2n$ values,

$$0, 1, 2, 3, \ldots 2n - 1,$$

before the variety of values of the trigonometrical expression into which it enters can be exhausted; and the same is true of $r - r'$. Now any of the n odd values of $r + r'$ may be combined with any of the n odd values of $r - r'$, and thus n^2 different values of the total expression are found. The even values of $r + r'$ and $r - r'$ produce as many more different values, hence the fractional differential coefficient of each of the quantities $\cos x$ and $\sin x$ has $2n^2$ different values.*

For example, the differential coefficients to the index $\frac{1}{2}$ of $\cos x$ and $\sin x$ have 2×2^2 or 8, values. Those of the former will be found to be

$$\pm\cos\left(\frac{\pi}{4} + x\right), \quad \pm\sqrt{(-1)}\cos\left(\frac{3\pi}{4} + x\right),$$
$$\pm\cos\left(\frac{5\pi}{4} + x\right), \quad \pm\sqrt{(-1)}\cos\left(\frac{7\pi}{4} + x\right).$$

The reader may observe that if differentiation to the index $\frac{1}{2}$ be performed twice in succession upon $\cos x$ by any one of

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* M. Liouville did not employ the direct process which we have, and consequently did not discover that the differential coefficients had more than $2n$ values.

these formulæ, provided the same be used both times, the result will always be $-\sin x$, as it should be.

7. The same multiplicity of values attends the differential coefficients of other trigonometrical functions. For instance, let us consider the functions $\varepsilon^{mx}\cos nx$ and $\varepsilon^{mx}\sin nx$. It will be convenient to assume $m = a\cos\theta$, and $n = a\sin\theta$. We may then change ax into x, and they become

$$\varepsilon^{x\cos\theta}\cos(x\sin\theta),\quad \varepsilon^{x\cos\theta}\sin(x\sin\theta),$$

which we shall call Θ and Θ' respectively, as Mr. R. Murphy [113] has done in the *Camb. Phil. Trans.*, vol. v., where he has proved some properties of them. We have then

$$\Theta + \sqrt{(-1)}\,\Theta' = \varepsilon^{x\cos\theta}\left\{\cos(x\sin\theta) + \sqrt{(-1)}\sin(x\sin\theta)\right\}$$
$$= \varepsilon^{x(\cos\theta+\sqrt{(-1)}\sin\theta)};\ \text{therefore}$$

$$\frac{d^a}{dx^a}\left\{\Theta + \sqrt{(-1)}\,\Theta'\right\} = \left\{\cos\theta + \sqrt{(-1)}\sin\theta\right\}^a \varepsilon^{x(\cos\theta+\sqrt{(-1)}\sin\theta)}$$
$$= \left\{\cos a(2r\pi+\theta) + \sqrt{(-1)}\sin a(2r\pi+\theta)\right\}$$
$$\varepsilon^{x\cos\theta}\left\{\cos(x\sin\theta) + \sqrt{(-1)}\sin(x\sin\theta)\right\}$$
$$= \varepsilon^{x\cos\theta}\left\{\cos(x\sin\theta + 2ar\pi + a\theta) + \sin(x\sin\theta + 2ar\pi + a\theta)\right\}.$$

In like manner

$$\frac{d^a}{dx^a}\left\{\Theta - \sqrt{(-1)}\,\Theta'\right\} =$$
$$\varepsilon^{x\cos\theta}\left\{\cos(x\sin\theta + 2ar'\pi + a\theta) - \sin(x\sin\theta + 2ar'\pi + a\theta)\right\}.$$

By addition and subtraction

$$\left.\begin{aligned}
&\frac{d^a\Theta}{dx^a} = \\
&\left\{\cos(r-r')\,a\pi + \sqrt{(-1)}\sin(r-r')\,a\pi\right\} \\
&\qquad \varepsilon^{x\cos\theta}\cos\left\{x\sin\theta + (r+r')\,a\pi + a\theta\right\}, \\
&\frac{d^a\Theta'}{dx^a} = \\
&\left\{\cos(r-r')\,a\pi + \sqrt{(-1)}\sin(r-r')\,a\pi\right\} \\
&\qquad \varepsilon^{x\cos\theta}\sin\left\{x\sin\theta + (r+r')\,a\pi + a\theta\right\},
\end{aligned}\right\}\ \ldots\ldots (V).$$

As before, the number of values of each of these expressions is twice the square of the denominator of a, supposed a rational fraction in its lowest terms.

The necessity of supposing r and r' different, may be proved as follows. If θ be changed into $-\theta$, Θ remains the same, and Θ' is altered only in sign: no new value of the differential coefficient ought therefore to be found by making

this change: but that would be the case if r and r' had been supposed equal; therefore that supposition would not give all the values of the differential coefficients. A similar remark applies to the last article.

8. We now come to another division of our subject, in which we shall prove certain formulæ discovered by M. Liouville, and applied by him to the solution of a variety of problems. The first of these is

$$\int_0^\infty \phi(x+\alpha)\,\alpha^{n-1}\,d\alpha = (-1)^n\,\Gamma(n)\,\frac{d^{-n}}{dx^{-n}}\,\phi(x)\;\ldots\ldots\;(W).$$

This may readily be proved by the theory of generating functions.

Let $G\phi(x)$ denote the generating function of $\phi(x)$, or [114] that function of t in the expansion of which the coefficient of t^x is $\phi(x)$; then we have

$$G\phi(x+\alpha) = t^{-\alpha}G\phi(x),$$

$$G\int_0^\infty \phi(x+\alpha)\,\alpha^{n-1}\,d\alpha = \int_0^\infty t^{-\alpha}\,\alpha^{n-1}\,d\alpha\,.\,G\phi(x).$$

Let $t^\alpha = \varepsilon^\theta$, then $\alpha = \dfrac{\theta}{\log t}$, and the second member of the preceding equation becomes

$$\Gamma(n)\,(\log t)^{-n}\,G\phi(x):$$

$$\text{but } \left(\log\frac{1}{t}\right)^r G\phi(x) = G\,\frac{d^r}{dx^r}\,\phi(x);$$

therefore $$G\int_0^\infty \phi(x+\alpha)\,\alpha^{n-1}\,d\alpha = (-1)^n\,\Gamma(n) G\,\frac{d^{-n}}{dx^{-n}}\,\phi(x),$$

and $$\int_0^\infty \phi(x+\alpha)\,\alpha^{n-1}\,d\alpha = (-1)^n\,\Gamma(n)\,\frac{d^{-n}}{dx^{-n}}\,\phi(x).$$

It may be remarked, that if the form of $\phi(x)$ be such that the first side is infinite, the second may be made to agree with it by the aid of the complementary function, if it do not without; so that the formula is true for all forms of $\phi(x)$.

9. An obvious use of this formula is to find the form of $\phi(x)$ when the value of the definite integral is known, as in the following problem.

Each particle M of a line AB, infinite in both directions, exerts upon a particle P without it a force perpendicular to the plane ABP, and proportional to sin PMA, and to an unknown function $F(r)$ of the distance PM or r. The total action of the line upon P is a known function $f(y)$ of the

perpendicular distance y of P from AB. It is required to find the function $F(r)$.

This is an important physical problem, for if AB represent a voltaic current, and P a pole of a magnet, the action of M on P is such as has been described; and the total action is found by experiment to be inversely as y.

Let C be the foot of the perpendicular from P on AB, and let $CM = z$, then $r^2 = y^2 + z^2$, and $\sin PMA = \dfrac{y}{r}$; whence the whole action of AB is

$$\int_{-\infty}^{+\infty} \frac{yF\sqrt{\{(y^2+z^2)\}}}{\sqrt{(y^2+z^2)}}\, dz = 2y \int_0^{\infty} \frac{F\sqrt{\{(y^2+z^2)\}}}{\sqrt{(y^2+z^2)}}\, dz,$$

which is equal to $f(y)$ by hypothesis.

[115] Assume $\dfrac{F(r)}{r} = \phi\,(r^2)$, therefore

$$2\int_0^{\infty} \phi\,(y^2+z^2)\, dz = \frac{f(y)}{y}.$$

Let $y^2 = x$, $z^2 = a$, then the first member becomes

$$\int_0^{\infty} \phi\,(x+a)\, a^{-\frac{1}{2}}\, da,$$

which by (W) is equal to

$$\sqrt{(-1)}\,\Gamma(\tfrac{1}{2})\left(\frac{d}{dx}\right)^{-\frac{1}{2}} \phi\,(x);$$

therefore $$\sqrt{(-1)}\,\Gamma(\tfrac{1}{2})\left(\frac{d}{dx}\right)^{-\frac{1}{2}} \phi\,(x) = \frac{f(\sqrt{x})}{\sqrt{x}},$$

and $$\phi\,(x) = \frac{1}{\sqrt{(-1)}\,\Gamma(\frac{1}{2})}\left(\frac{d}{dx}\right)^{\frac{1}{2}} \frac{f(\sqrt{x})}{\sqrt{x}}.$$

Supposing that the differentiation is performed, and r^2 is substituted for x, $F(r)$, which is equal to $r\phi\,(r^2)$, is known.

If $f(y) = \dfrac{a}{y}$, $\dfrac{f(\sqrt{x})}{\sqrt{x}} = \dfrac{a}{x}$, and by formula ($E$), p. 14, we find

$$\phi\,(x) = \tfrac{1}{2}\, a x^{-\frac{3}{2}};$$

therefore $$F(r) = \frac{a}{2r^2}.$$

It is easy to verify this result by ordinary integration, and thus to confirm the truth of the principles of the new calculus.

10. It will be convenient to possess a formula in which the limits of the integral shall be 0 and 1. Such a one may

be obtained from (W) by changing the independent variable.

Let $x + a = \dfrac{x}{\theta}$; then when $a = 0$, $\theta = 1$, and when $a = \infty$, $\theta = 0$: also, $a = \dfrac{1-\theta}{\theta}\,.\,x$, $da = -\dfrac{x d\theta}{\theta^2}$, and the formula is changed into

$$\int_0^1 \phi\left(\frac{x}{\theta}\right)\frac{x^n}{\theta^{n+1}}(1-\theta)^{n-1}\,d\theta = (-1)^n\,\Gamma(n)\,\frac{d^{-n}}{dx^{-n}}\,\phi(x).$$

Let $x^{n+1}\,\phi(x) = \psi\left(\dfrac{1}{x}\right)$; therefore

$$\int_0^1 \psi\left(\frac{\theta}{x}\right)(1-\theta)^{n-1}\,d\theta = (-1)^n\,\Gamma(n)\,x\,\frac{d^{-n}}{dx^{-n}}\left\{\frac{1}{x^{n+1}}\,\psi\left(\frac{1}{x}\right)\right\}\ldots(X).$$

The following modification of the formula is often convenient.

Assume $\theta = \beta x$ and $x = \dfrac{1}{y}$, then we have

$$\int_0^y \psi(\beta)\,(y-\beta)^{n-1}d\beta = (-1)^n\Gamma(n)\,y^{n-1}\,d^{-n}\left\{y^{n+1}\psi(y)\right\}\left(d\,.\,\frac{1}{y}\right)^n\ldots(Y).$$

The differentiation indicated in the second member [116]
must be performed with respect to $\dfrac{1}{y}$ as the independent variable.

11. As an example of the use of the last formula, we select the following problem, which includes, as a particular case, that of finding the tantochronous curve.

To find the curve such that the time occupied by the descent of a particle sliding along it by the action of gravity to a given point from a given vertical height above it, shall be a given function of that height.

Let x be measured vertically. Then the known expression for the time of descent from a height h to the origin, is

$$\int_0^h \frac{1}{\sqrt{\{2g\,(h-x)\}}}\,\frac{ds}{dx}\,dx;$$

but by hypothesis this is equal to the known function $f(h)$. Putting $\dfrac{ds}{dx} = \psi(x)$, and comparing the integral with that in formula (Y), we find

$$\sqrt{(-1)}\,\Gamma\left(\tfrac{1}{2}\right)h^{-\frac{1}{2}}\,d^{-\frac{1}{2}}\left\{h^{\frac{3}{2}}\,\psi(h)\right\}\left(d\,.\,\frac{1}{h}\right)^{\frac{1}{2}} = \sqrt{(2g)}\,f(h),$$

whence, observing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\psi(h) = -\sqrt{(-1)}\sqrt{\frac{2g}{\pi}} \cdot h^{-\frac{3}{2}} d^{\frac{1}{2}} \{\sqrt{h} f(h)\} d.\left(\frac{1}{h}\right)^{-\frac{1}{2}}.$$

By changing h into x, $\psi(x)$ is known, and thus we have a differential equation to the curve.

Suppose, as a particular case, that $f(h) = ch^n$; then, after changing h into $\frac{1}{z}$ for convenience of operation, we have

$$\psi\left(\frac{1}{z}\right) = -\sqrt{(-1)}\, c\sqrt{\frac{2g}{\pi}} \cdot z^{\frac{3}{2}} \left(\frac{d}{dz}\right)^{\frac{1}{2}} z^{-n-\frac{1}{2}}$$

$$= c\sqrt{\frac{2g}{\pi}} \cdot \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} z^{-n+\frac{1}{2}}$$

by (F), if n be positive. Therefore the equation to the curve is

$$\frac{ds}{dx} = c\sqrt{\frac{2g}{\pi}} \cdot \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} x^{n-\frac{1}{2}}.$$

If $n = 0$, or the time be independent of the height, we have

$$\frac{ds}{dx} = c\sqrt{\left(\frac{2g}{\pi}\right)} x^{-\frac{1}{2}},$$

which belongs to a cycloid.

[117] If $n = \frac{1}{2}$, or the time vary as the square root of the height,

$$\frac{ds}{dx} = c\sqrt{\frac{g}{2}},$$

which belongs to a straight line.

If $n = 1$, or the time vary as the height,

$$\frac{ds}{dx} = c\,\frac{2\sqrt{(2g)}}{\pi} x^{\frac{1}{2}}:$$

putting $c\,\frac{2\sqrt{(2g)}}{\pi} = \frac{1}{\sqrt{a}}$, and solving the equation, we obtain

$$y = \tfrac{2}{3}\,\frac{(x-a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

We might give many more instances of the use of these formulæ, but our limits will not allow us: and therefore we recommend to such of our readers as are particularly interested in this subject, the original Memoirs above referred to, which are more complete in the applications of the Calculus than in its principles.

S. S. G.

ON THE TANGENT PLANE OF THE ELLIPSOID.*

In an article which appeared in the first Number of this Journal, some examples were given of the advantage of employing, in certain cases, a form of the equation to the tangent of the ellipse, which does not involve the coordinates of the point of contact. There is an analogous form of the equation to the tangent plane to the ellipsoid, viz.

$$lx + my + nz = \sqrt{(l^2a^2 + m^2b^2 + n^2c^2)},$$

l, m, n being the cosines of the angles which a perpendicular on the tangent plane makes with the axes. We shall not give a proof of this, but refer the reader to Hymers' *Geometry of Three Dimensions*, Art. 29. A few examples of its application may perhaps be useful.

1. To find the locus of the intersection of three tangent planes to an ellipsoid, which are mutually at right angles.

Let (lmn), $(l'm'n')$, $(l''m''n'')$ be the cosines of the angles which perpendiculars on the planes make with the [118]
axes. Their equations will be

$$lx + my + nz = \sqrt{(l^2a^2 + m^2b^2 + n^2c^2)},$$

$$l'x + m'y + n'z = \sqrt{(l'^2a^2 + m'^2b^2 + n'^2c^2)},$$

$$l''x + m''y + n''z = \sqrt{(l''^2a^2 + m''^2b^2 + n''^2c^2}.$$

Squaring both sides of each of these equations, adding, and observing that, since the planes are at right angles,

$$l^2 + l'^2 + l''^2 = 1,\quad m^2 + m'^2 + m''^2 = 1,\quad n^2 + n'^2 + n''^2 = 1,$$

$$lm + l'm' + l''m'' = 0,\ ln + l'n' + l''n'' = 0,\ mn + m'n' + m''n'' = 0,$$

we have $$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

This proposition was first proved by Monge.

2. To find the locus of intersection of the perpendicular on the tangent plane with that plane.

The equation to the tangent plane being

$$lx + my + nz = \sqrt{(l^2a^2 + m^2b^2 + n^2c^2)} \ldots\ldots\ldots (1),$$

the equations to the perpendicular will be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Multiplying the successive terms on both sides by these equal quantities respectively, we have

$$x^2 + y^2 + z^2 = \sqrt{(a^2x^2 + b^2y^2 + c^2z^2)},$$

* From a Correspondent.

or $$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

3. Three planes, mutually at right angles, touch three spheres whose radii are a, a', a'', respectively. To find the locus of intersection of the planes.

In this case the equations will be

$$lx + my + nz = a,$$
$$l'x + m'y + n'z = a',$$
$$l''x + m''y + n''z = a''.$$

Squaring, and adding, we shall have

$$x^2 + y^2 + z^2 = a^2 + a'^2 + a''^2.$$

4. Three planes, mutually at right angles, touch three concentric ellipsoids, whose principal sections have their foci coincident. To find the locus of intersection of these planes.

Let abc, $a'b'c'$, $a''b''c''$, be the semiaxes of the three ellipsoids; then the equations will be

$$lx + my + nz = \sqrt{(l^2a^2 + m^2b^2 + n^2c^2)}, \qquad [119]$$
$$l'x + m'y + n'z = \sqrt{(l'^2a'^2 + m'^2b'^2 + n'^2c'^2)},$$
$$l''x + m''y + n''z = \sqrt{(l''^2a''^2 + m''^2b''^2 + n''^2c''^2)}.$$

Squaring both sides, and adding, we have on the first side $x^2 + y^2 + z^2$, as before. Also,

$$l^2a^2 + m^2b^2 + n^2c^2 = a^2 + m^2(b^2 - a^2) + n^2(c^2 - a^2),$$
$$l'^2a'^2 + m'^2b'^2 + n'^2c'^2 = b'^2 + l'^2(a'^2 - b'^2) + n'^2(c'^2 - b'^2),$$

which, since the foci in the principal sections coincide,

$$= b'^2 + l'^2(a^2 - b^2) + n'^2(c^2 - b^2).$$

Similarly,

$$l''^2a''^2 + m''^2b''^2 + n''^2c''^2 = c''^2 + l''^2(a^2 - c^2) + m''^2(b^2 - c^2).$$

Therefore the second side becomes, by addition,

$$\begin{aligned} &= a^2 + b'^2 + c''^2 \\ &+ a^2(l'^2 + l''^2 - m^2 - n^2) \\ &+ b^2(m^2 + m''^2 - l'^2 - n'^2) \\ &+ c^2(n^2 + n'^2 - l''^2 - m''^2). \end{aligned}$$

But $\quad l^2 + l'^2 + l''^2 = 1 = l^2 + m^2 + n^2; \therefore l'^2 + l''^2 = m^2 + n^2.$

Similarly, $\quad m^2 + m''^2 = l'^2 + n'^2$ and $n^2 + n'^2 = l''^2 + m''^2$.

Hence the equation to the locus is

$$x^2 + y^2 + z^2 = a^2 + b'^2 + c''^2.$$

Again, since $\quad a^2 - b^2 = a'^2 - b'^2,$

or $\quad a^2 + b'^2 = a'^2 + b^2,$

and $\quad b^2 + c''^2 = b''^2 + c^2;$

therefore, by addition,

$$a^2 + b'^2 + c''^2 = a'^2 + b''^2 + c^2.$$

And, in the same way,

$$a^2 + b'^2 + c''^2 = a''^2 + b^2 + c'^2.$$

Consequently the equation may be put into the form

$$x^2 + y^2 + z^2 = \tfrac{1}{3}\{(a^2 + b^2 + c^2) + (a'^2 + b'^2 + c'^2) + (a''^2 + b''^2 + c''^2)\}$$

This result was first published in the 19th volume of the *Annales de Mathématiques.*

If the equation to the surface be

$$x = \frac{y^2}{a} + \frac{z^2}{a'},$$

the equation to the tangent plane may be exhibited in the form

$$l\,(lx + my + nz) + \tfrac{1}{4}\,(am^2 + a'n^2) = 0.$$

As there is no proof of this in Hymers' *Analytical Geometry,* we subjoin one.

Let the equation to the tangent plane be [120]

$$lx + my + nz = \delta.$$

This must be identical with the common equation to the tangent plane drawn at a point $(x'y'z')$, viz.

$$x + x' = 2\left(\frac{y'y}{a} + \frac{z'z}{a'}\right),$$

or

$$\frac{2y'y}{ax'} + \frac{2z'z}{a'x'} - \frac{x}{x'} = 1;$$

therefore we must have

$$\frac{l}{\delta} = -\frac{1}{x'}, \text{ or } x' = -\frac{\delta}{l},$$

$$\frac{m}{\delta} = \frac{2y'}{ax'} = -\frac{2ly'}{a\delta},$$

whence

$$y' = -\frac{ma}{2l}.$$

Similarly,

$$z' = -\frac{na'}{2l}.$$

But

$$x' = \frac{y'^2}{a} + \frac{z'^2}{a'}:$$

hence

$$-\frac{\delta}{l} = \frac{m^2a}{4l^2} + \frac{n^2a'}{4l^2},$$

or

$$\delta = -\frac{1}{4l}(m^2a + n^2a').$$

Consequently the equation is

$$l(lx + my + nz) + \tfrac{1}{4}(m^2a + n^2a') = 0.$$

The reader will find no difficulty in proving, by means of this equation, that tangent planes, mutually at right angles, will always intersect in a plane whose equation is

$$x + \tfrac{1}{4}(a + a') = 0;$$

and that the locus of intersection of the tangent plane and the perpendicular upon it from the vertex, is a surface, represented by the equation

$$x(x^2 + y^2 + z^2) + \tfrac{1}{4}(ay^2 + a'z^2) = 0.$$

Q.

[121]

TRANSFORMATION FROM RECTANGULAR TO POLAR COORDINATES IN DIFFERENTIAL EXPRESSIONS.

When we have a differential expression involving two independent variables, which we wish to transform into one with two other independent variables, the method to be pursued is not at first sight obvious. If, for instance, we have the double integral

$$\iint V\,dx\,dy$$

where V is a function of x and y, and we wish to transform it so as to involve r and θ, the four quantities x, y, r, θ, being connected by the equations

$$x = f(r, \theta), \quad y = F(r, \theta),$$

so that $$dx = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta, \quad dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta,$$

we cannot multiply the expressions for dx and dy together as they stand; because in the integral y is supposed to be constant when x varies, and x to be constant when y varies. We must introduce one of these conditions by supposing $dx = 0$ when y varies, or $dy = 0$ when x varies.

Taking the first of these, we have the equations

$$0 = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta,$$

$$dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta.$$

Eliminating $d\theta$ between these, we find

$$\frac{dx}{d\theta}\,dy = \left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right) dr.$$

From this it follows, that when $dy = 0$, $dr = 0$. Hence

$$dx = \frac{dx}{d\theta}\,d\theta.$$

Substituting these values in the double integral, it becomes

$$\iint V\left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right) dr\, d\theta,$$

where V involves only r and θ.

When x and y are the rectangular, r, θ the polar coordinates of a point, $\quad x = r\sin\theta,\ y = r\cos\theta,$

and $\quad \iint V\,dx\,dy = \iint V\,rdr\,d\theta.$

If there be three independent variables the same method is to be pursued; but as we have to eliminate between three equations it becomes very complicated, even though we avail ourselves of the method of elimination by cross multiplication given in our first Number. In the particular case of [122] the quantities representing the coordinates in space of a point, the assumption of a subsidiary quantity much facilitates the calculation. For let $\quad \iiint V\,dx\,dy\,dz$

be the function to be transformed, and

$$x = r\cos\theta,\ y = r\sin\theta\sin\phi,\ z = r\sin\theta\cos\phi.$$

Assume $\rho = r\sin\theta$. Then we have

$$y = \rho\sin\phi,\ z = \rho\cos\phi \ldots\ldots\ldots\ldots(1),$$

$$\rho = r\sin\theta,\ x = r\cos\theta \ldots\ldots\ldots\ldots(2).$$

We can then transform, as we did before, from x, y, z, to x, ρ, ϕ, and after that from x, ρ, ϕ, to r, θ, ϕ.

The simplicity of the method consists in this, that the second operation is exactly similar to the first, so that we do not require to repeat the calculation, but merely to write down the result, substituting r for ρ, and θ for ϕ. Effecting the first operation,

$$dx\,dy\,dz = dx\,d\rho\,d\phi\left\{\frac{dy}{d\phi}\frac{dz}{d\rho} - \frac{dy}{d\rho}\frac{dz}{d\phi}\right\},$$

which by equations (1) becomes

$$dx\,dy\,dz = \rho\,dx\,d\rho\,d\phi.$$

Again, changing from x, ρ, ϕ, to r, θ, ϕ, we find

$$dx\,dy\,dz = \rho r\,dr\,d\theta\,d\phi = r^2\,dr\sin\theta\,d\theta\,d\phi.$$

Therefore $\quad \iiint V\,dx\,dy\,dz = \iiint V\,r^2\,dr\sin\theta\,d\theta\,d\phi.$

The same assumption may be usefully applied to transforming the expression

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$$

from rectangular to polar coordinates; for it will be seen, that having found $\frac{dV}{dy}$ and $\frac{d^2V}{dy^2}$, the whole expression may be written down without further trouble.

Thus we find

$$\frac{dV}{dy}=\sin\phi\,\frac{dV}{d\rho}+\frac{\cos\phi}{\rho}\,\frac{dV}{d\phi},$$

$$\frac{d^2V}{dy^2}=\sin^2\phi\,\frac{d^2V}{d\rho^2}+\frac{\cos^2\phi}{\rho^2}\,\frac{d^2V}{d\phi^2}+\frac{\cos^2\phi}{\rho}\,\frac{dV}{d\rho}$$

$$+\frac{2\sin\phi\cos\phi}{\rho^2}\left(\rho\,\frac{d^2V}{d\rho\,d\phi}-\frac{dV}{d\phi}\right).$$

The expression for $\frac{d^2V}{dz^2}$ is got by putting $90-\phi$ for ϕ in the above, so that without writing it down

$$\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=\frac{d^2V}{d\rho^2}+\frac{1}{\rho^2}\,\frac{d^2V}{d\phi^2}+\frac{1}{\rho}\,\frac{dV}{d\rho}.$$

[123] Putting r for ρ and θ for ϕ, we have the similar expression

$$\frac{d^2V}{d\rho^2}+\frac{d^2V}{dx^2}=\frac{d^2V}{dr^2}+\frac{1}{r^2}\,\frac{d^2V}{d\theta^2}+\frac{1}{r}\,\frac{dV}{dr}.$$

And the expression for $\frac{dV}{d\rho}$ in r and θ being similar to that for $\frac{dV}{dy}$ in r and ϕ, gives

$$\frac{1}{\rho}\,\frac{dV}{d\rho}=\frac{1}{r}\,\frac{dV}{dr}+\frac{\cot\theta}{r^2}\,\frac{dV}{d\theta}.$$

Adding these three expressions,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=\frac{1}{r^2}\left(\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\,\frac{dV}{d\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\,\frac{d^2V}{d\phi^2}+\frac{d^2V}{dr^2}+\frac{2}{r}\,\frac{dV}{dr}$$

$$=\frac{1}{r^2}\,\frac{d}{d\cos\theta}\left(\sin^2\theta\,\frac{dV}{d\cos\theta}\right)+\frac{1}{r^2\sin^2\theta}\,\frac{d^2V}{d\phi^2}+\frac{1}{r}\,\frac{d^2(rV)}{dr^2},$$

which is the well-known expression.

A. S.

ON THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS.

THE integration of Partial Differential Equations is much facilitated by the principle which we have developed in our preceding Numbers, and it is the more remarkable, that it has been so little applied to these equations, as the first step was taken many years ago. Fourier, in his *Traité de Chaleur*, published in 1822, has shewn that the series which are obtained in the solution of several partial differential equations, may be conveniently expressed by the separation of the symbols of operation from those of quantity. But though he has used this method very frequently, yet he appears to have had some unwillingness to give himself up to it entirely as a guide in his investigations, as if he were not familiar with the principles on which it is founded. His idea apparently was, that the expression which he obtained as solutions might be conveniently expressed by separating the symbols of operation, and not that the symbolical expressions are the proper solutions of the equations, and the series merely the expansion of them. Other French writers seem to have avoided carefully entering at all on the track which Fourier opened: Poisson in particular, in [124] a digression on the subject of partial differential equations in the second volume of his *Mécanique*, does not put in the symbolical form the solution of a very simple equation, which is so given by Fourier. Mr. Greatheed, in a paper published in the number of the *Philosophical Magazine* for September 1837, was, we believe, the first to call the attention of mathematicians to the utility of this method in the case of partial differential equations, but he had not then reduced it to its greatest degree of simplicity; and his paper is chiefly occupied with a particular class of equations of the first order with variable coefficients, which are not so interesting as many others with constant coefficients. We shall here, therefore, proceed to give several examples of the application of the principles which we laid down in Art. v. of our first Number.

And, first, we may observe generally, that linear partial differential equations between any number of variables with constant coefficients, are to be treated exactly like ordinary differential equations with regard to one of the variables, the symbols of operation of the others being treated as constants.

If, for instance, we have the equation

$$a\frac{dz}{dx} + b\frac{dz}{dy} = c.$$

This may be put under the form

$$\left(a\frac{d}{dx} + b\frac{d}{dy}\right)z = c,$$

and therefore $\quad z = \left(a\frac{d}{dx} + b\frac{d}{dy}\right)^{-1}(c + 0),$

where we suppose x to be the variable, and $\dfrac{d}{dy}$ a constant with regard to it. Now the operation $\left(a\dfrac{d}{dx} + b\dfrac{d}{dy}\right)^{-1}$ by the theorem given in page 25, is equivalent to

$$a^{-1}\,\varepsilon^{-\frac{b}{a}x\frac{d}{dy}}\int dx\,\varepsilon^{\frac{b}{a}x\frac{d}{dy}},$$

which being performed, gives

$$z = \left(\frac{d}{dy}\right)^{-1}\frac{c}{b} + \frac{1}{a}\,\varepsilon^{-\frac{b}{a}x\frac{d}{dy}}f(y),$$

$f(y)$ being an arbitrary function of y, taking the place of the constant in ordinary differential equations.

From this expression we get

$$z = \frac{cy}{b} + f(ay - bx),$$

[125] as by Taylor's theorem

$$\varepsilon^{h\frac{d}{dx}}f(x) = f(x + h).$$

We might equally well have supposed y to be the variable, and $\dfrac{d}{dx}$ a constant with regard to it.

Again, taking the well-known equation for the motion of waves,

$$\frac{d^2z}{dt^2} - a^2\frac{d^2z}{dx^2} = 0,$$

it may be put under the form

$$\left(\frac{d^2}{dt^2} - a^2\frac{d^2}{dx^2}\right)z = 0;$$

and integrating it like the ordinary differential equation

$$\left(\frac{d^2}{dt^2} - n^2\right)z = 0,$$

we find $$z = \epsilon^{at\frac{d}{dx}}\phi(x) + \epsilon^{-at\frac{d}{dx}}\psi(x),$$
or $$z = \phi(x+at) + \psi(x-at).$$

The equation $r - 2a\,s + a^2t = 0$ may be put under the form

$$\left(\frac{d}{dx} - a\,\frac{d}{dy}\right)^2 z = 0,$$

the solution of which is, by the theorem in page 25,

$$z = \epsilon^{ax\frac{d}{dy}} \int^2 dx^2 . 0$$
$$= \epsilon^{ax\frac{d}{dy}} \{x\phi(y) + \psi(y)\},$$

$\phi(y)$ and $\psi(y)$ being arbitrary functions of y arising from the integration. Hence, finally, we have

$$z = x\phi(y+ax) + \psi(y+ax).$$

The equation $r - a^2t + 2abp + 2a^2bq = 0$
is equivalent to

$$\left\{\frac{d}{dx} - \left(a\,\frac{d}{dy} - 2ab\right)\right\}\left(\frac{d}{dx} + a\,\frac{d}{dy}\right) z = 0.$$

Integrating with regard to the first factor, we have

$$\left(\frac{d}{dx} + a\,\frac{d}{dy}\right) z = \epsilon^{x\left(a\frac{d}{dy} - 2ab\right)}\phi(y).$$

Integrating with regard to the remaining factor,

$$z = \epsilon^{-ax\frac{d}{dy}} \int dx\, \epsilon^{2ax\left(\frac{d}{dy} - b\right)} \phi(y) + \epsilon^{-ax\frac{d}{dy}}\psi(y) \qquad [126]$$
$$= \epsilon^{-ax\frac{d}{dy}} \epsilon^{2ax\left(\frac{d}{dy} - b\right)} \left\{2a\left(\frac{d}{dy} - b\right)\right\}^{-1} \phi(y) + \psi(y - ax)$$
$$= \epsilon^{ax\frac{d}{dy}} \epsilon^{-2abx} \epsilon^{by} \phi_1(y) + \psi(y - ax),$$

by changing the arbitrary function

$$= \epsilon^{-2abx} \epsilon^{b(y+ax)} \phi_1(y+ax) + \psi(y-ax)$$
$$= \epsilon^{b(y-ax)} \phi_1(y+ax) + \psi(y-ax).$$

Let us take also the equation

$$r - c^2t = xy.$$

Without operating on each factor separately, we may arrive more readily at the result by the same means as those employed in page 28 of our first Number.

For we have

$$z = \left(\frac{d^2}{dx^2} - c^2 \frac{d^2}{dy^2}\right)^{-1} (xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \frac{d^{-2}}{dx^{-2}} \left(1 - c^2 \frac{d^2}{dy^2} \frac{d^{-2}}{dx^{-2}}\right)^{-1} (xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \left(1 - c^2 \frac{d^2}{dy^2} \frac{d^{-2}}{dx^{-2}}\right)^{-1} \left(\frac{x^3 y}{6}\right) + \phi(y + ax) + \psi(y - ax).$$

Therefore $z = \dfrac{x^3 y}{6} + \phi(y + ax) + \psi(y - ax),$

the arbitrary functions being added as in the second example.

A very simple equation, being one which occurs in the theory of heat, is

$$\frac{dv}{dt} = a \frac{d^2 v}{dx^2},$$

which is the expression for the rectilinear propagation of heat. The solution of this is easily seen to be, if we integrate with regard to t,

$$v = \varepsilon^{at \frac{d^2}{dx^2}} f(x),$$

$f(x)$ being an arbitrary function. If the sign of operation be expanded, we shall obtain a series which is the solution derived by Poisson from the method of indeterminate coefficients. Laplace has deduced from the series an elegant expression for the solution under the form of a definite integral; but it may be more easily deduced from the symbolical solution in the following manner.

Since $$\int_{-\infty}^{\infty} \varepsilon^{-\omega^2} d\omega = \sqrt{\pi},$$

and also $$\int_{-\infty}^{\infty} \varepsilon^{-(\omega - b)^2} d\omega = \sqrt{\pi},$$

[127] we can put the expression for v under the form

$$\sqrt{\pi}\, v = \int_{-\infty}^{\infty} d\omega\, \varepsilon^{at \frac{d^2}{dx^2}} \varepsilon^{-\left(\omega - \sqrt{(at)} \frac{d}{dx}\right)^2} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega\, \varepsilon^{-\omega^2} . \varepsilon^{2\omega \sqrt{(at)} \frac{d}{dx}} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega\, \varepsilon^{-\omega^2} f\{x + 2\omega \sqrt{(at)}\}$$

by Taylor's theorem ; and this is Laplace's expression.

This equation $\frac{dv}{dt} = a\frac{d^2v}{dx^2}$ not being homogeneous in the index of the operations, admits of two solutions of very different characters. The one, which we have found by integrating with regard to t, contains only one arbitrary function of x. The other, which may be found by integration with regard to x, must contain two arbitrary functions of t, as the index of operation is of the second degree. If we write the equation in the form

$$\frac{d^2v}{dx^2} - \frac{1}{a}\frac{dv}{dt} = 0,$$

and integrate by the method employed in page 28, we find for the integral

$$v = \left(x + \frac{1}{a}\frac{x^3}{1.2.3}\frac{d}{dt} + \frac{1}{a^2}\frac{x^5}{1.2.3.4.5}\frac{d^2}{dt^2} + \&c.\right)\phi(t)$$
$$+\left(1 + \frac{1}{a}\frac{x^2}{1.2}\frac{d}{dt} + \frac{1}{a^2}\frac{x^4}{1.2.3.4}\frac{d^2}{dt^2} + \&c.\right)\psi(t).$$

It seems at first sight anomalous, that the same equation should have two solutions so different in character: the following is the explanation of the difficulty. Since by Maclaurin's theorem any function of a variable may be expressed by means of its differential coefficients taken with regard to that variable, for the particular value 0 of the variable, we know the function if we can determine its successive differential coefficients. Now from the equation $\frac{dv}{dt} = a\frac{d^2v}{dx^2}$ we can, when we know the value of v when $t = 0$, determine the values of all the differential coefficients with regard to t when $t = 0$. So that in the resulting expression deduced from Maclaurin's theorem there is only one quantity left undetermined, which is the arbitrary function $f(x)$, introduced in the integration. But from the equation $\frac{d^2v}{dx^2} = \frac{1}{a}\frac{dv}{dt}$, we can only determine from the value of v when $x = 0$ the values of the alternate differential coefficients with regard to x. There must consequently be introduced another undetermined quantity, namely, the value of $\frac{dv}{dx}$ when $x = 0$; for knowing these two quantities we can determine the values of all the successive differential coefficients with regard to x. [128]

The equation for determining the vibratory motion of an elastic spring is

$$\frac{d^2v}{dt^2} + \frac{d^4v}{dx^4} = 0.$$

The solution of which is readily seen to be

$$v = \cos\left(t\frac{d^2}{dx^2}\right)F(x) + \sin\left(t\frac{d^2}{dx}\right)f(x).$$

The equation for determining the vibratory motion of elastic plates is

$$\frac{d^2v}{dt^2} + \frac{d^4v}{dx^4} + 2\frac{d^4v}{dx^2dy^2} + \frac{d^4v}{dy^4} = 0,$$

which may be put under the form

$$\frac{d^2v}{dt^2} + \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2 v = 0;$$

the integral of which is

$$v = \cos t\left(\frac{d^2}{dx} + \frac{d^2}{dy^2}\right)F(x, y) + \sin t\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)f(x, y).$$

In investigating the motion of heat in a ring, we obtain as an equation for determining the temperature at any time and point,

$$\frac{dv}{dt} = k\frac{d^2v}{dx^2} - hv;$$

the solution of which is

$$v = \varepsilon^{\left(k\frac{d^2}{dx^2}-h\right)t} f(x)$$

$$= \varepsilon^{-ht}\varepsilon^{kt\frac{d^2}{dx^2}} f(x),$$

an expression closely connected with one previously given.

In the examples which we have given, the coefficients are constant; but if one of the variables only enters into the coefficients, and we integrate with regard to the other, as the one variable is unaffected by the sign of operation with regard to the other, it may be considered as a constant in the integration.

A good example of this kind of equation is that which expresses the motion of heat in a solid cylinder of infinite length, namely

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right),$$

the solution of which is

$$v = \varepsilon^{at\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)} f(x),$$

or, as it may be expressed, [129]

$$v = \varepsilon^{at\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)} f(x).$$

But this will only be possible when the integration is of the first degree, since, as we shewed in our first Number,

the symbols of operation become subject to different laws when the variable itself and the sign of differentiation are both involved. And this leads us to the consideration of equations with variable coefficients. As in the case of ordinary differential equations, the solution of this class is attended with great difficulties, so as to become almost impossible for equations of an order higher than the first, it is to be supposed that these difficulties are no way diminished in the case of partial differential equations. Of these however, when of the first order, Mr. Greatheed has shewn, that a large class may be solved like ordinary differential equations. It is included under the form

$$\frac{dz}{dx} + XY\frac{dz}{dy} = Pz + Q,$$

where X is a function of x only, Y of y only, and P and Q of both x and y. This may be reduced to the form

$$\frac{dz}{dx} + X\frac{dz}{dy} = Pz + Q,$$

by a change of the variable; for if $y' = \int \frac{dy}{Y}$, then $Y\frac{dz}{dy} = \frac{dz}{dy'}$; so that it is only necessary to consider the latter equation. Let it be put under the form

$$\frac{dz}{dx} + \left(X\frac{d}{dy} - P\right)z = Q,$$

and treated as an ordinary linear differential equation between z and x. If we integrate by the method of integrating factors, the factor is $\varepsilon^{\int dx\left(X\frac{d}{dy} - P\right)}$, and the equation becomes

$$\frac{d}{dx}\left\{\varepsilon^{\int dx\left(X\frac{d}{dy} - P\right)}.\,z\right\} = \varepsilon^{\int dx\left(X\frac{d}{dy} - P\right)}.\,Q;$$

whence

$$z = \varepsilon^{-\int dx\left(X\frac{d}{dy} - P\right)}\int dx\left\{\varepsilon^{\int dx\left(X\frac{d}{dy} - P\right)}.Q\right\} + \varepsilon^{-\int dx\left(X\frac{d}{dy} - P\right)}.\,\phi(y),$$

$\phi(y)$ being an arbitrary function of y.

We shall take as an example the equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} = nz.$$

Let $\frac{dy}{y} = dt$, and therefore $y = \varepsilon^t$. Then

$$\frac{dz}{dx} + \frac{1}{x}\frac{dz}{dt} = n\frac{z}{x}.$$

[130] The integral of which, by the preceding formula, is

$$z = \varepsilon^{-\int dx\left(\frac{1}{x}\frac{d}{dt}-\frac{n}{x}\right)}\phi(t),$$

or $$z = \varepsilon^{n\log x}\,.\,\varepsilon^{\log x\frac{d}{dt}}\phi(t)$$
$$= x^n\,\phi(t-\log x) = x^n\,\phi(\log y - \log x) = x^n\,\psi\left(\frac{y}{x}\right).$$

The equation $$x\frac{dz}{dx} - y\frac{dz}{dy} = \frac{x^2}{y}$$

may be integrated in the same way. Or if we change both the independent variables, making $\frac{dy}{y} = dt$ and $\frac{dx}{x} = du$, it becomes

$$\left(\frac{d}{du} - \frac{d}{dt}\right)z = \varepsilon^{2u}\,\varepsilon^{-t},$$

which gives $$z = \varepsilon^{u\frac{d}{dt}}\int du\,\varepsilon^{-u\frac{d}{dt}}\,\varepsilon^{2u}\,\varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}}\phi(t)$$
$$= \varepsilon^{u\frac{d}{dt}}\int du\,\varepsilon^{3u}\,\varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}}\phi(t)$$
$$= \frac{\varepsilon^{2u}\,\varepsilon^{-t}}{3} + \phi(t+u)$$
$$= \frac{x^2y}{3} + \psi(xy),$$

as $$u = \log x,\ t = \log y.$$

The equation $$y\frac{dz}{dx} + x\frac{dz}{dy} = z$$

may, by changing $y\,dy$ into $\frac{1}{2}\,d.y^2$, be transformed into

$$\frac{dz}{dx} + \left(2x\frac{d}{d.y^2} - \frac{1}{y}\right)z = 0.$$

The integral of which is

$$z = \varepsilon^{-\int dx\left(2x\frac{d}{d.y^2}-\frac{1}{y}\right)}\phi(y^2)$$
$$= \varepsilon^{\int\frac{dx}{y}}\,\varepsilon^{-x^2\frac{d}{d.y^2}}\phi(y^2)$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}}\left\{\varepsilon^{x^2\frac{d}{d.y^2}}\left(\varepsilon^{\int\frac{dx}{y}}\right)\phi(y^2)\right\}$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}}\left\{\varepsilon^{\int\frac{dx}{\sqrt{(y^2+x^2)}}}\phi(y^2)\right\}$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}}\left\{\varepsilon^{\log[x+\sqrt{(x^2+y^2)}]}\,.\,\phi(y^2)\right\}$$
$$= {}^{-x^2\frac{d}{d.y^2}}\left[\{x+\sqrt{(x^2+y^2)}\}\,\phi(y^2)\right]$$
$$= (x+y)\,\phi(y^2-x^2).$$

Other equations, which at first sight do not appear [131] to come under this form, may be reduced to it by a proper assumption of a new independent variable. For instance, the equation

$$(x+y)\frac{dz}{dx}+(y-x)\frac{dz}{dy}=z,$$

is converted into

$$u\frac{dz}{dx}-2x\frac{dz}{du}=z,$$

by supposing $y=u-x$. These however are particular cases which come under no general rule.

D. F. G.

ON STABLE AND UNSTABLE EQUILIBRIUM.*

LET P, P', P''.... be any number of forces acting upon a material system, and let dp, dp', dp''.... be the virtual velocities of their points of application, estimated positive in the directions in which the forces act; then, when P, P', P''... are in equilibrium, the equilibrium is *stable* when

$$\int(Pdp+P'dp'+\ldots.)$$

is a *maximum;* and *unstable* when this is *not a maximum:* and *vice versâ.*

We shall demonstrate this in the case of one rigid body; and the same reasoning can easily be applied when the system contains several rigid bodies.

When P, P', P''...... act upon a rigid body, they can always be reduced to two forces, but not to one. Let R and R' be these forces, and dr, dr' the virtual velocities of their points of application; then

$$Pdp+P'dp'+\ldots.=Rdr+R'dr',$$

$$\therefore\quad u=\int(Pdp+P'dp'+\ldots.)=\int(Rdr+R'dr').$$

Hence $$du=Rdr+R'dr',$$

and $$d^2u=Rd^2r+dRdr+R'd^2r'+dR'dr'.$$

Now when P, P', P''.... are in equilibrium,

$$Pdp+P'dp'+P''dp''+\ldots.=0,$$

$$\therefore\quad Rdr+R'dr'=0,\quad\therefore\quad du=0;$$

* From a Correspondent.

and, since dr and dr' are independent of each other,

$$R = 0 \text{ and } R' = 0,$$

$$\therefore \quad d^2u = dR.dr + dR'.dr'.$$

When u is a maximum, d^2u is negative, and therefore (since dr and dr' are independent of each other) $dR.dr$ and [132] $dR'.dr'$ are both negative. Hence if dr be positive, dR is negative; and *vice versâ:* and so with dr' and dR'.

From this we learn, that if the body receive any slight displacement from the situation of equilibrium, two small forces dR and dR' are brought into play, which act opposite to the directions in which the points, at which they act, have moved in consequence of the displacement of the body. Hence the effect of these forces is to draw the body back towards its position of equilibrium; and therefore the equilibrium was *stable*.

Conversely, in order that the equilibrium may be stable, the forces dR, dR' put in play by the displacement must act so as to carry the body back to its situation of equilibrium; and therefore the virtual velocities dr and dr' must have different signs to dR and dR' respectively: hence d^2u is negative, and therefore u a maximum.

If u be a *minimum*, we may shew in the same manner that the equilibrium is *unstable*, and *vice versâ*.

If u be *neither a maximum nor a minimum*, then d^2u (or d^3u, if d^2u vanishes) will be positive for some displacements, and negative for others; and will therefore be apparently *dubious*, though in fact it will be *unstable:* for although a displacement may be found which shall cause the body to return to its position of rest, yet in oscillating through the position of rest it may come to one, from which it will not return to its position of rest. The converse of this is also true.

Hence, if by *stable* equilibrium we mean, that for all small displacements the body oscillates about its position of rest, then the equilibrium is *stable* when u is a *maximum*, and in no other case: in all other cases the equilibrium is *unstable*.

N.B. We have supposed that $Pdp + P'dp' + \ldots\ldots$ is a perfect differential. If this be not the case, the above reasoning is still true; but the result should be stated thus: when the equilibrium is *stable*, the first differential of $Pdp + P'dp' + \ldots\ldots$ (or if that and the second vanish, the third) is *negative*, and *vice versâ:* in all other cases the equilibrium is *unstable*.

If the system consist of several bodies, and R, R', R'', R''' ......be the smallest number of forces to which the system of forces acting on the bodies can be reduced, then

$$du = Rdr + R'dr' + R''dr'' + R'''dr''' + \ldots\ldots$$

in which dr, dr', dr'', dr'''....are independent of each other: and the reasoning upon these and the forces R, R', R'', R''' ......will be as above.

The following is an application of this principle to fluid equilibrium.

Problem. Required the form of equilibrium of a mass of incompressible fluid, every particle of which is attracted towards a centre of force, varying as a function of the distance; and to determine whether the equilibrium be *stable* or *unstable*.

Let the fluid be referred to polar coordinates, the [133] centre of force being the origin, θ the angle which r makes with a fixed line through the origin, and ω the angle which the plane through r and the fixed line makes with a fixed plane passing through the fixed line: and let (as usual) $\cos\theta = \mu$. It is evident that the fluid must completely surround the centre of force: let r and r' be the radii of the external and internal surfaces; these are functions of μ and ω. The volume of the fluid is given: let c be the radius of the sphere which has the same volume; hence

$$\frac{4\pi}{3}c^3 = \int_{-1}^{1}\int_{0}^{2\pi}\int_{\mathrm{r}'}^{\mathrm{r}} r^2\, d\mu\, d\omega\, dr = \tfrac{1}{3}\int_{-1}^{1}\int_{0}^{2\pi}(\mathrm{r}^3 - \mathrm{r}'^3)\, d\mu\, d\omega.$$

Since r^3 and r'^3 are functions of μ and ω, they may be expanded in series of Laplace's coefficients: let

$$\mathrm{r}^3 = a^3 + \alpha(Y_1 + Y_2 + \ldots\ldots) = a^3 + \alpha y,$$

and $$\mathrm{r}'^3 = a'^3 + \alpha(Y_1' + Y_2' + \ldots\ldots) = a'^3 + \alpha y',$$

where α is a numerical coefficient, the use of which will be seen hereafter.

Then, by a fundamental property of these functions,

$$\int_{-1}^{1}\int_{0}^{2\pi} y\, d\mu\, d\omega = 0;$$

and similarly of y'. Hence we have, by the above equation,

$$c^3 = a^3 - a'^3.$$

Let $f(r)$ be the force of attraction on a unit of mass at a distance r from the centre of force; then $f(r)\,\delta m$ is the

attraction on the element of the mass δm: this attraction acts in the line r, and tends to shorten r: hence, in this case,

$$\dot{u} = \int \Sigma . \{-f(r)\, \delta m\}\, dr \text{; and this} = \Sigma . \int \{-f(r)\, dr\}\, \delta m.$$

The symbol of integration Σ refers to the differential of the mass δm. Now δm equals (as above, if density = 1)

$$r^2 \sin\theta\, d\theta\, d\omega\, dr = -\, r^2\, d\mu\, d\omega\, dr,$$

and the limits of r, ω, μ are respectively r' and r, 0, and 2π, 1 and -1: then, putting

$$\int -f(r)\, dr = f_1(r), \quad \text{and } \int r^2 f_1(r)\, dr = \phi\,(r^3),$$

we have

$$u = \int_{-1}^{1}\int_0^{2\pi}\int_{\mathrm{r}'}^{\mathrm{r}} r^2 f_1(r)\, d\mu\, d\omega\, dr = \int_{-1}^{1}\int_0^{2\pi} \{\phi\,(\mathrm{r}^3) - \phi\,(\mathrm{r}'^3)\}\, d\mu\, d\omega.$$

But $\phi\,(\mathrm{r}^3) - \phi\,(\mathrm{r}'^3) = \phi\,(a^3 + \alpha y) - \phi\,(a'^3 + \alpha y')$

$$= \phi\,(a^3) - \phi\,(a'^3) + \alpha y\, \phi'\,(a^3) - \alpha y'\, \phi'\,(a'^3)$$
$$+ \tfrac{1}{2}\, \alpha^2 y^2\, \phi''\,(a^3) - \tfrac{1}{2}\, \alpha^2 y'^2\, \phi''\,(a'^3) + \ldots.$$

in which the accents to ϕ denote differentiation with respect to a^3 and a'^3. Hence

$$u = 4\pi\phi\,(a^3) - 4\pi\phi\,(a'^3)$$
$$+ \tfrac{1}{2}\, \alpha^2 \int_{-1}^{1}\int_0^{2\pi} \{y^2 \phi''(a^3) - y'^2 \phi''\,(a'^3)\}\, d\mu\, d\omega + \ldots.$$

[134] It is very easily shewn, that $\phi''(a^3) = -\dfrac{1}{9a^2} f(a)$, and is therefore *negative* in every case of *attraction*, (but positive if the force were repulsive).

From these calculations we gather the following results. If α be a very small quantity, the increment of u above what it becomes when $\alpha = 0$ (that is, when $\mathrm{r} = a$ and $\mathrm{r}' = a'$) does not involve the first power of α: hence, when $\mathrm{r} = a$ and $\mathrm{r}' = a'$, we have $du = 0$, and therefore the fluid is in equilibrium.

Hence a form of equilibrium is that of a hollow sphere of any dimensions (so long as the *volume* be constant), and having the centres of the bounding surfaces at the centre of force.

And the equilibrium of the *external* surface is *stable*, because for all values of y when $y' = 0$ the multiplier of α^2 is negative, and therefore u is a *maximum*: and the equilibrium of the *internal* surface is *unstable*, because for all values of y' when $y = 0$ the multiplier of α^2 is positive, and therefore u is a *minimum*. On the whole, the equilibrium is dubious.

Cor. 1. If $a' = 0$, then $y' = 0$, and the equilibrium is altogether stable.

COR. 2. If the force were *repulsive*, and the fluid were held in equilibrium by a rigid spherical envelope; then $y = 0$, and the multiplier of a^2 would be negative, since $\phi''(a'^3)$ is then positive, and u is a *maximum*, and the equilibrium of the internal surface is *stable*.

J. H. P.

ANALYTICAL GEOMETRY OF THREE DIMENSIONS. NO. II.

WE proceed to give some instances of the advantage of symmetry in applying the Differential Calculus to Geometry of three dimensions. The following form of the equation to the tangent plane is probably known to many of our readers; but, as Leroy has only deduced it from that in terms of the partial differential coefficients of one of the coordinates with respect to the other two, we will give an independent and easy proof of it.

1. If $F(x, y, z) = 0$, be the equation to a surface, the locus of the tangent lines drawn to it at a point (x, y, z) is a plane whose equation is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0 \ldots .(1).$$

The equation to any line drawn through the point [135] x, y, z, is

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} \ldots\ldots\ldots\ldots (2).$$

This line will in general be cut by the surface in one or more other points. Let the coordinates of the nearest of these be x_1, y_1, z_1, and let each member of equation (2), when x', y', z' become x_1, y_1, z_1, be assumed equal to r, then

$$x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr:$$

but $$F(x_1, y_1, z_1) = 0;$$

therefore $$F(x + lr,\ y + mr,\ z + nr) = 0.$$

Expanding, and observing that $F(x, y, z) = 0$,

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} + \frac{r}{2}\left(l^2\frac{d^2F}{dx^2} + \&c.\right) + \ldots . = 0.$$

When the line becomes a tangent, the points (x, y, z), (x_1, y_1, z_1), approach infinitely near to one another, and r

becomes indefinitely small; therefore, taking the limit of the preceding equation,

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0.$$

Multiplying the several terms of this by the several members of equation (2), we obtain

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0,$$

for the equation to the locus of the tangent lines.

2. Since the normal line is perpendicular to the tangent plane, its equations are

$$\frac{x' - x}{\frac{dF}{dx}} = \frac{y' - y}{\frac{dF}{dy}} = \frac{z' - z}{\frac{dF}{dz}} \quad \ldots\ldots\ldots\ldots (3).$$

3. We shall next investigate an expression in terms of the partial differential coefficients of $F(x, y, z)$, for the radius of curvature of a section of a surface made by any plane passing through the normal at a given point.

Let $\frac{dF}{dx} = U$, $\frac{dF}{dy} = V$, $\frac{dF}{dz} = W$, then the equations to the normal are

$$\frac{x' - x}{U} = \frac{y' - y}{V} = \frac{z' - z}{W} \quad \ldots\ldots\ldots\ldots (4).$$

Again, if ds be an element of the section, and dx, dy, dz its projections, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles [136] which it makes with the axes, and therefore the equation to the normal plane to the section is

$$(x' - x)\,dx + (y' - y)\,dy + (z' - z)\,dz = 0 \;\ldots\; (5).$$

The line of intersection of two consecutive normal planes will be determined by (5) and its differential, which is

$$(x' - x)\,d^2x + (y' - y)\,d^2y + (z' - z)\,d^2z - ds^2 = 0 \;\ldots\; (6);$$

and the intersection of this line with the normal to the surface will evidently be the centre of curvature of the section. We may suppose x', y', z' to belong to this point; they will then be the same in (4) as in (5) and (6): and assuming each member of (4) equal to Q, and substituting for x', y', z' in (6), we have

$$(U\,d^2x + V\,d^2y + W\,d^2z)\,Q - ds^2 = 0.$$

But if ρ be the radius of curvature,

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2 = (U^2 + V^2 + W^2)\, Q^2,$$

therefore $$\rho = \pm \frac{\sqrt{(U^2 + V^2 + W^2)} \,.\, ds^2}{U d^2x + V d^2y + W d^2z} \,\ldots\ldots\ldots\, (7).$$

This expression may be transformed as follows. Since

$$U dx + V dy + W dz = 0,$$

differentiating, and assuming $\frac{d^2F}{dx^2} = u$, $\frac{d^2F}{dy^2} = v$, $\frac{d^2F}{dz^2} = w$, $\frac{d^2F}{dy\,dz} = u'$, $\frac{d^2F}{dz\,dx} = v'$, $\frac{d^2F}{dx\,dy} = w'$,

$$U d^2x + V d^2y + W d^2z + u\,dx^2 + v\,dy^2 + w\,dz^2$$
$$+ 2u'\,dy\,dz + 2v'\,dz\,dx + 2w'\,dx\,dy = 0.$$

Employing this equation, and assuming $\frac{dx}{ds} = l$, $\frac{dy}{ds} = m$, $\frac{dz}{ds} = n$, (7) is changed into

$$\rho = \mp \frac{\sqrt{(U^2 + V^2 + W^2)}}{l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw'} \,\ldots\, (8).$$

4. It may be as well here to shew how formulæ involving the partial differential coefficients of F, may readily be transformed into others in terms of the partial differential coefficients of one of the coordinates as z. Suppose the equation to be put under the form $f(x, y) - z = 0$, so that $F(x, y, z) = f(x, y) - z$. Then

$$\left.\begin{array}{l} \frac{dF}{dx} = \frac{d}{dx} f(x, y) = \frac{dz}{dx}, \quad \frac{dF}{dy} = \frac{dz}{dy}, \quad \frac{dF}{dz} = -1, \\ \frac{d^2F}{dx^2} = \frac{d^2z}{dx^2}, \quad \frac{d^2F}{dy^2} = \frac{d^2z}{dy^2}, \quad \frac{d^2F}{dz^2} = 0, \\ \frac{d^2F}{dy\,dz} = 0, \quad \frac{d^2F}{dz\,dx} = 0, \quad \frac{d^2F}{dx\,dy} = \frac{d^2z}{dx\,dy}, \end{array}\right\} \ldots (9).$$

If we substitute these values in (8) and adopt the usual notation, we obtain the known expression [137]

$$\rho = \mp \frac{\sqrt{(1 + p^2 + q^2)}}{l^2 r + 2lm\, s + m^2 t}.$$

5. To find the sections whose radius of curvature is a *maximum* or *minimum*, and the values of those radii.

In formula (8) let $\mp\sqrt{(U^2+V^2+W^2)} = P$; then

$$\frac{P}{\rho} = l^2u + m^2v + n^2w + 2mnu' + 2nlv' + 2lmw' \; \dots \; (10),$$

the quantities l, m, n being connected by the two equations

$$l^2 + m^2 + n^2 = 1 \; \dots\dots\dots\dots \; (11),$$

$$lU + mV + nW = 0 \; \dots\dots\dots\dots \; (12).$$

Differentiating these equations,

$$(lu + mw' + nv')\,dl + (lw' + mv + nu')\,dm + (lv' + mu' + nw)\,dn = 0 \; \dots\dots \; (13),$$

$$ldl + mdm + ndn = 0 \; \dots\dots\dots \; (14),$$

$$Udl + Vdm + Wdn = 0 \; \dots\dots\dots \; (15),$$

(13) + (14) λ + (15) μ gives, on equating to zero, the coefficients of each differential,

$$\left.\begin{aligned} lu + mw' + nv' + \lambda l + \mu U &= 0 \\ lw' + mv + nu' + \lambda m + \mu V &= 0 \\ lv' + mu' + nw + \lambda n + \mu W &= 0 \end{aligned}\right\} \; \dots\dots \; (16).$$

Multiplying the first of these by l, the second by m, and the third by n, adding, and reducing by (10), (11), (12),

$$\frac{P}{\rho} + \lambda = 0.$$

Substituting for λ in (16), they become

$$\left.\begin{aligned} \left(u - \frac{P}{\rho}\right)l + w'm + v'n + \mu U &= 0 \\ w'l + \left(v - \frac{P}{\rho}\right)m + u'n + \mu V &= 0 \\ v'l + u'm + \left(w - \frac{P}{\rho}\right)n + \mu W &= 0 \end{aligned}\right\} \; \dots \; (17).$$

To obtain the equation for ρ, it remains to eliminate l, m, n, μ from equations (17) and (12). The result is

[138] $$U^2\left(v - \frac{P}{\rho}\right)\left(w - \frac{P}{\rho}\right) + V^2\left(w - \frac{P}{\rho}\right)\left(u - \frac{P}{\rho}\right) + W^2\left(u - \frac{P}{\rho}\right)\left(v - \frac{P}{\rho}\right)$$
$$- 2VWu'\left(u - \frac{P}{\rho}\right) - 2WUv'\left(v - \frac{P}{\rho}\right) - 2UVw'\left(w - \frac{P}{\rho}\right)$$
$$- U^2u'^2 - V^2v'^2 - W^2w'^2$$
$$+ 2VWv'w' + 2WUw'u' + 2UVu'v'$$
$$= 0 \; \dots\dots\dots\dots\dots\dots \; (18).$$

6. If the equation to the surface be of the form

$$\phi(x) + \chi(y) + \psi(z) = 0,$$

which includes, among others, all the surfaces of the second order when referred to their principal diametral planes; U, V, W are functions of x, y, z alone, respectively, so that u', v', w' are all zero, and the equation (18) simplifies considerably. In short, in this case, equations (17) give immediately

$$l = \frac{\mu U}{u - \frac{P}{\rho}}, \quad m = \frac{\mu V}{v - \frac{P}{\rho}}, \quad n = \frac{\mu W}{w - \frac{P}{\rho}};$$

and, by substituting these values in (12), we have

$$\frac{U^2}{u - \frac{P}{\rho}} + \frac{V^2}{v - \frac{P}{\rho}} + \frac{W^2}{w - \frac{P}{\rho}} = 0 \;\ldots\ldots\ldots\; (19).$$

7. EXAMPLE. To find the principal radii of curvature of an ellipsoid.

In this case

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0,$$

therefore

$$U = \frac{2x}{a^2}, \quad V = \frac{2y}{b^2}, \quad W = \frac{2z}{c^2}, \quad u = \frac{2}{a^2}, \quad v = \frac{2}{b^2}, \quad w = \frac{2}{c^2},$$

and the value of P is

$$2\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)} = \frac{2}{p},$$

if p denote the perpendicular from the centre on the tangent plane. Substituting these values in (19), we have

$$\frac{x^2}{a^2(p\rho - a^2)} + \frac{y^2}{b^2(p\rho - b^2)} + \frac{z^2}{c^2(p\rho - c^2)} = 0,$$

which may easily be put in the form

$$\rho^2 - \{a^2 + b^2 + c^2 - (x^2 + y^2 + z^2)\}\frac{\rho}{p} + \frac{a^2b^2c^2}{p^4} = 0.$$

8. As another example, we propose to find the prin- [139]
cipal radii of curvature at any point of the surface whose equation is $a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2 \ldots\ldots\ldots (a)$, and which is the locus of the extremity of the perpendicular from the centre on the tangent plane to an ellipsoid. Put

$$x^2 + y^2 + z^2 = r^2;$$

then $\qquad F(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 - r^4,$

$$U = 2x(a^2 - 2r^2), \quad V = 2y(b^2 - 2r^2), \quad W = 2z(c^2 - 2r^2),$$
$$u = 2(a^2 - 2r^2 - 4x^2),\ v = 2(b^2 - 2r^2 - 4y^2),\ w = 2(c^2 - 2r^2 - 4z^2),$$
$$u' = -8yz, \quad v' = -8zx, \quad w' = -8xy.$$

Instead of substituting these values in equation (18), we shall employ the equations from which that is deduced, namely (12) and (17), because by this method it is more easy to obtain a result of a simple form. Of these, (12) becomes

$$lx(a^2 - 2r^2) + my(b^2 - 2r^2) + nz(c^2 - 2r^2) = 0 \ \ldots (b),$$

and the first of (17),

$$l\left(a^2 - 2r^2 - 4x^2 - \frac{P}{\rho}\right) - 4mxy - 4nxz + \mu x(a^2 - 2r^2) = 0,$$

or $l\left(a^2 - 2r^2 - \dfrac{P}{\rho}\right) - 4x(lx + my + nz) + \mu x(a^2 - 2r^2) = 0.$

Assume $\qquad lx + my + nz = k \ \ldots\ldots\ldots\ldots (c);$

then we have

$$\left.\begin{aligned} l\left(a^2 - 2r^2 - \frac{P}{\rho}\right) - 4kx + \mu x(a^2 - 2r^2) = 0 \\ m\left(b^2 - 2r^2 - \frac{P}{\rho}\right) - 4ky + \mu y(b^2 - 2r^2) = 0 \\ n\left(c^2 - 2r^2 - \frac{P}{\rho}\right) - 4kz + \mu z(c^2 - 2r^2) = 0 \end{aligned}\right\} \ldots (d),$$

and we shall obtain the equation for ρ by eliminating l, m, n, k, μ, from the five equations $(b), (c), (d)$. For this purpose, multiply equations (d) by x, y, z respectively, and add them, reducing by $(a), (b), (c)$; then we find

$$k\left(\frac{P}{\rho} + 4r^2\right) + \mu r^4 = 0.$$

By this relation, eliminate μ from equations (d), and we have

$$l = \frac{4(a^2 - r^2)r^2 + (a^2 - 2r^2)\dfrac{P}{\rho}}{a^2 - 2r^2 - \dfrac{P}{\rho}} \cdot \frac{xk}{r^4},$$

with corresponding expressions for m and n. Substituting them in (b), and denoting

$$F(a, x) + F(b, y) + F(c, z) \text{ by } \Sigma F(a, x),$$

[140]

$$\Sigma \frac{4(a^2 - r^2)(a^2 - 2r^2) r^2 + (a^2 - 2r^2)^2 \dfrac{P}{\rho}}{a^2 - 2r^2 - \dfrac{P}{\rho}} . x^2 = 0;$$

but by (a) $\quad \Sigma (a^2 - r^2) x^2 = 0$:
multiplying the last equation by $4r^2$, subtracting from the preceding, and dividing the result by $\dfrac{P}{\rho}$,

$$\Sigma \frac{a^4 x^2}{a^2 - 2r^2 - \dfrac{P}{\rho}} = 0,$$

which is the simplest form in which the equation can be pre sented. The value of P will be found to be

$$2 \sqrt{(a^4x^2 + b^4y^2 + c^4z^2)}.$$

The length of the perpendicular from the centre on the tangent plane to this surface is

$$\frac{r^4}{\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)}},$$

therefore if p denote that line, $P = \dfrac{2r^4}{p}$.

9. The directions of the sections of greatest and least curvature will be found by eliminating λ and μ from equations (16). The result is

$$(lu + mw' + nv')(mW - nV) + (lw' + mv + nu')(nU - lW) \\ + (lv' + mu' + nw')(lV - mU) = 0,$$

or $$(Vv' - Ww')l^2 + (Ww' - Uu')m^2 + (Uu' - Vv')n^2 \\ + \{Wv' - Vw' + U(v - w)\} mn + \{Uw' - Wu' + V(w - u)\} nl \\ + \{Vu' - Uv' + W(u - v)\} lm = 0 \ldots (20),$$

which equation, together with (11) and (12), determines the values of l, m, n. It is easy to see that there will be two sets of values of l^2, m^2, n^2, and no more.

10. To prove that the directions of greatest and least curvature are at right angles.

For abbreviation, write the equation (20)

$$Ll^2 + Mm^2 + Nn^2 + L'mn + M'nl + N'lm = 0.$$

Substituting the value of n derived from equation (12),

$$(Ll^2 + Mm^2 + N'lm) W^2 - (L'm + M'l)(lU + mV) W \\ + N(lU + mV)^2 = 0,$$

or $(MW^2+NV^2-L'VW)\,m^2 + \{2NUV - W(L'U+M'V)\}\,lm$
$$+ (NU^2 + LW^2 - M'WU)\,l^2 = 0,$$

[141] a quadratic equation in $\frac{m}{l}$; whence we find, that if l_1, l_2 are the two values of l, and m_1, m_2 those of m,

$$\frac{m_1 m_2}{l_1 l_2} = \frac{NU^2 + LW^2 - M'WU}{MW^2 + NV^2 - L'VW};$$

so that if we assume

$$l_1 l_2 = K(MW^2 + NV^2 - L'VW),$$

we shall have

$$m_1 m_2 = K(NU^2 + LW^2 - M'WU),$$

and also $\quad n_1 n_2 = K(LV^2 + MU^2 - N'UV).$

Hence, the cosine of the angle between the two directions,

or $l_1 l_2 + m_1 m_2 + n_1 n_2 = K\{L(V^2 + W^2) + M(W^2 + U^2)$
$$+ N(U^2 + V^2) - L'VW - M'WU - N'UV\}.$$

The value of the quantity within the brackets is zero, as may easily be seen by recurring to equation (20) for the values of LM, &c. and collecting the terms multiplied by u', v', &c. respectively. Therefore the two directions are at right angles.

11. To find the lines of curvature of a given surface, or the direction in which we must pass from one point to a consecutive, in order that the normals at the two points may meet.

Retaining the same notation, the equations to the normal at a point x, y, z are

$$\frac{x' - x}{U} = \frac{y' - y}{V} = \frac{z' - z}{W}.$$

Let each member be assumed equal to Q, then

$$x' = x + QU, \quad y' = y + QV, \quad z' = z + QW.$$

If x', y', z' belong to the point in which two consecutive normals meet, the differentials of these three equations must be verified at the same time with them, therefore

$$\left.\begin{aligned} dx + QdU + UdQ &= 0, \\ dy + QdV + VdQ &= 0, \\ dz + QdW + WdQ &= 0, \end{aligned}\right\} \;\ldots\ldots\ldots(21),$$

or
$$u\,dx + w'dy + v'dz + \frac{1}{Q}\,dx + U\,\frac{dQ}{Q} = 0,$$
$$w'dx + v\,dy + u'dz + \frac{1}{Q}\,dy + V\,\frac{dQ}{Q} = 0,$$
$$v'dx + u'dy + w\,dz + \frac{1}{Q}\,dz + W\,\frac{dQ}{Q} = 0.$$

These equations are of the same form as equations (16), therefore the elimination of Q and dQ from these must give the same relation between $dx, dy, dz,$ as the elimination of λ and μ from (16) gave between $l, m, n,$ namely (20). And if
$$dx^2 + dy^2 + dz^2 = ds^2,$$
$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ will be the cosines of the angles which an [142]
element of a line of curvature makes with the axes, and they satisfy the same system of equations as $l, m, n,$ namely (11), (12), (20). Therefore the directions in which two consecutive normals meet coincide with those of greatest and least curvature.

For the sake of shortness, we shall eliminate Q and dQ from equations (21), instead of their expanded forms. This is easily done by cross-multiplication, and the result is
$$U(dV dz - dW dy) + V(dW dx - dU dz) + W(dU dy - dV dx) = 0 \ldots (22):$$
from this equation and that to the surface, the differential to the projection of a line of curvature on any coordinate plane may be found.

12. Example. To find the lines of curvature of an ellipsoid. Let the equation be
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$
then
$$U = \frac{2x}{a^2}, \quad V = \frac{2y}{b^2}, \quad W = \frac{2z}{c^2},$$
$$dU = \frac{2}{a^2}\,dx, \quad dV = \frac{2}{b^2}\,dy, \quad dW = \frac{2}{c^2}\,dz.$$
Substituting in (22),
$$(b^2 - c^2)\,x\,dy\,dz + (c^2 - a^2)\,y\,dz\,dx + (a^2 - b^2)\,z\,dx\,dy = 0.^*$$

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* For other methods of treating this equation, see vol. II. p. 133, vol. III. p. 264, and vol. IV. p. 279.

Multiply by $\frac{z}{c^2}$, and substitute the values

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{z\,dz}{c^2} = -\frac{x\,dx}{a^2} - \frac{y\,dy}{b^2},$$

therefore

$$\{(b^2 - c^2)\,x\,dy + (c^2 - a^2)\,y\,dx\}\left(\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2}\right)$$
$$- (a^2 - b^2)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy = 0,$$

or, reducing,
$$\frac{b^2 - c^2}{b^2}\,xy\,dy^2$$
$$+ \left\{\frac{a^2 - c^2}{a^2}\,x^2 - \frac{b^2 - c^2}{b^2}\,y^2 - (a^2 - b^2)\right\} dx\,dy - \frac{a^2 - c^2}{a^2}\,xy\,dx^2 = 0.$$

To integrate this, multiply by $4xy$, and let $x^2 = u$, $y^2 = v$, then

$$\frac{b^2 - c^2}{b^2}\,u\,dv^2$$
$$+ \left\{\frac{a^2 - c^2}{a^2}\,u - \frac{b^2 - c^2}{b^2}\,v - (a^2 - b^2)\right\} du\,dv - \frac{a^2 - c^2}{a^2}\,v\,du^2 = 0,$$

or $$\left(\frac{b^2 - c^2}{b^2}\,dv + \frac{a^2 - c^2}{a^2}\,du\right)(u\,dv - v\,du) - (a^2 - b^2)\,du\,dv = 0,$$

[143]

or
$$u - v\frac{du}{dv} = \frac{(a^2 - b^2)\frac{du}{dv}}{\frac{b^2 - c^2}{b^2} + \frac{a^2 - c^2}{a^2}\frac{du}{dv}},$$

an equation of Clairaut's form, whose solution is

$$u - Cv = \frac{C(a^2 - b^2)}{\frac{b^2 - c^2}{b^2} + C\frac{a^2 - c^2}{a^2}},$$

or
$$x^2 - Cy^2 = \frac{C(a^2 - b^2)}{\frac{b^2 - c^2}{b^2} + C\frac{a^2 - c^2}{a^2}},$$

the equation to an ellipse or hyperbola. For the method of determining the constant, see *Leroy*, No. 422.

S. S. G.

MATHEMATICAL NOTES.

1. *Taylor's Theorem.*—The following proof of Taylor's Theorem by means of the separation of symbols of operation from those of quantity, may be interesting to some of our readers. Let there be an operation, which we may for convenience call D, which performed h times on $f(x)$ converts it into $f(x+h)$; that is, let $D^h f(x) = f(x+h)$, we wish to determine the nature of D. Now we know that

$$\frac{d}{dx} f(x) = \frac{f(x+h) - f(x)}{h}$$

when h is indefinitely small; therefore

$$\frac{d}{dx} f(x) = \frac{D^h f(x) - f(x)}{h} = \frac{D^h - 1}{h} f(x)$$

when h is indefinitely small. Therefore, taking the symbols of operation separate,

$$\frac{d}{dx} = \frac{D^h - 1}{h} \text{ when } h = 0,$$

which gives $$D = \left(1 + h\frac{d}{dx}\right)^{\frac{1}{h}} \text{ when } h = 0.$$

Expanding by the Binomial Theorem, and making $h = 0$, we find

$$D = 1 + \frac{d}{dx} + \frac{1}{1.2}\frac{d^2}{dx^2} + \frac{1}{1.2.3}\frac{d^3}{dx^3} + \&c.$$

or $$D = \varepsilon^{\frac{d}{dx}}; \qquad [144]$$

therefore $$f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x),$$
which is Taylor's Theorem.

2. *Problem from the Papers of* 1835.—If p, r be the perpendicular from the origin on the tangent plane and the radius vector of any surface, then $\frac{p^2}{r}$ will be the perpendicular on the tangent plane at the corresponding point of the surface, which is the locus of the extremity of p.

Let x, y, z be the coordinates of the first surface, α, β, γ of the second. As p is the perpendicular on the tangent plane, the equation to the plane is

$$\alpha x + \beta y + \gamma z = p^2 = \alpha^2 + \beta^2 + \gamma^2 \; \ldots\ldots \; (1).$$

Since this plane is a tangent to the surface, this equation will hold good if for x, y, z we put $x + dx$, $y + dy$, $z + dz$. Whence $$\alpha dx + \beta dy + \gamma dz = 0 \; \ldots\ldots\ldots\ldots\ldots \; (2).$$

Now if $V=0$ be the equation to the surface of which α, β, γ are the coordinates, and P be the perpendicular on the tangent plane; then

$$P = \frac{\alpha \frac{dV}{d\alpha} + \beta \frac{dV}{d\beta} + \gamma \frac{dV}{d\gamma}}{\left\{\left(\frac{dV}{d\alpha}\right)^2 + \left(\frac{dV}{d\beta}\right)^2 + \left(\frac{dV}{d\gamma}\right)^2\right\}^{\frac{1}{2}}};$$

also, $$\frac{dV}{d\alpha} d\alpha + \frac{dV}{d\beta} d\beta + \frac{dV}{d\gamma} d\gamma = 0 \ldots\ldots\ldots (3).$$

Now differentiating (1), considering $\alpha, \beta, \gamma, x, y, z$ as variables, and paying regard to (2), we have

$$(x - 2\alpha)\, d\alpha + (y - 2\beta)\, d\beta + (z - 2\gamma)\, d\gamma = 0 \ldots. (4),$$

$\lambda\,(3) - (4)$ gives, on equating to 0, the coefficients of each of the differentials

$$\lambda \frac{dV}{d\alpha} = x - 2\alpha, \quad \lambda \frac{dV}{d\beta} = y - 2\beta, \quad \lambda \frac{dV}{d\gamma} = z - 2\gamma.$$

Substituting these in the expression for P, it becomes

$$P = \frac{2\,(\alpha^2 + \beta^2 + \gamma^2) - (\alpha x + \beta y + \gamma z)}{\{x^2 + y^2 + z^2 + 4\,[\alpha^2 + \beta^2 + \gamma^2 - (\alpha x + \beta y + \gamma z)]\}^{\frac{1}{2}}},$$

which by (1) is reduced to

$$P = \frac{\alpha^2 + \beta^2 + \gamma^2}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{p^2}{r}.$$

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[145] ON THE RESIDUAL CALCULUS.

Cauchy, in several articles in his "*Exercices*," has developed a new Calculus, to which he has given the name of the "*Calcul des Résidus*." Few persons appear to have followed the author in the use of it; and in our own language I do not recollect having met even with an allusion to it: yet it deserves some attention at the hands of mathematicians, were it only from the deserved celebrity of its author. I propose to give here merely a slight sketch of its principles, with a few examples; and if any one should be induced from this specimen to wish for a more complete knowledge of this Calculus, I must refer him to the Memoirs of its author.

The Residual Calculus bears a certain analogy to the Differential Calculus, for in both the object is to investigate the nature and properties of certain functions which appear in an indeterminate form, but yet have finite values.

Let $f(x)$ be a function of x, which becomes infinite when $x = a$; that is, let a be a root of the equation

$$f(x) = \infty, \text{ or } \frac{1}{f(x)} = 0 \text{ .... (1)}.$$

Then the expression $(x - a)f(x)$ becomes indeterminate when $x = a$, as it takes the form $0 \times \infty$, though it may still have really a finite value. Suppose this to be the case, and that it is a certain function of a which we may call $R(a)$. Then we have

$$(x - a)f(x) = R(x) \text{ when } x = a \text{ ....(2)},$$

which gives
$$f(x) = \frac{R(x)}{x - a} \text{ when } x = a \text{ ....(3)}.$$

This function $R(a)$ is called the residue of $f(x)$ [146] with respect to a. The operation of finding the residue is called the extraction of the residue, and is represented by the characteristic symbol $\mathcal{E}$, so that we write

$$R(a) = \mathcal{E} f(x).$$

In equation (3) let us suppose $x = a + h$. Then

$$f(a + h) = \frac{R(a + h)}{h} = \frac{1}{h}\{R(a) + h\,R'(a) + \&\text{c.}\},$$

when $x = a$, or $h = 0$.

Whence we find the residue $R(a)$ of $f(x)$ is the coefficient of $\frac{1}{h}$ in the expansion of $f(a + h)$, where a is a root of the equation $f(x) = \infty$.

We have spoken only of one root of this equation, but there may be several, and for each of these there is a corresponding residue. The sum of the residues corresponding to each root of the equation is called the *integral* residue of $f(x)$. To distinguish the integral from the partial residues, I shall suffix the root to the Residual Symbol in the latter case.* Thus, $\mathcal{E} f(x)$ will represent the integral residue of $f(x)$, $\mathcal{E}_a f(x)$, $\mathcal{E}_b f(x)$, $\mathcal{E}_c f(x)$, &c. will represent the partial residues with respect to a, b, c, &c., these being roots of the equation $f(x) = \infty$.

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* This is not Cauchy's notation.

Let us suppose, now, that instead of the expression

$$(x - a) f(x)$$

having a finite value when $x = a$, that it is the expression

$$(x - a)^m f(x)$$

which has a finite value when $x = a$; in which case $f(x) = \infty$ is said to have m equal roots. If, as before, we suppose $R(a)$ to be the finite value, we shall have

$$f(x) = \frac{R(x)}{(x-a)^m} \text{ when } x = a,$$

or putting $x = a + h$, and expanding $R(a + h)$ by Taylor's Theorem,

$$f(a+h) = \frac{1}{h^m}\left\{R(a) + \frac{h}{1} R'(a) + \&\text{c.} + \frac{h^{(m-1)} R^{(m-1)}(a)}{1.2\ldots(m-1)} + \&\text{c.}\right\}.$$

The coefficient of $\frac{1}{h}$ in this expansion is

$$\frac{R^{(m-1)}(a)}{1.2\ldots(m-1)},$$

which is the residue of the function $f(x)$ corresponding to $x = a$.

[147] Since $\quad R(x) = h^m f(x) = h^m f(a + h),$

$$R^{(m-1)}(a) = \frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0;$$

and therefore we have

$$(4)\ \mathcal{E}_a f(x) = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0.$$

I shall now proceed to shew how the extraction of residues may be simplified when $f(x)$ has particular forms: but previously it is necessary to explain the following notation. When $f(x)$ consists of distinct factors, which are functions of x, the roots of $f(x) = \infty$ may be the roots of any one factor equated to ∞.

When then we wish to indicate that the extraction of residues has reference only to the roots of one of the factors, that factor is enclosed in brackets [ ]. As, for instance, if $f(x) = \phi(x) . \psi(x)$, and we wish to represent the extraction of the residues with reference to the roots of $\phi(x) = \infty$, we shall write

$$\mathcal{E}\,[\phi(x)] . \psi(x).$$

If a factor is of the form $\frac{1}{\phi(x)}$, we should properly represent the extraction of the residues with respect to it by

$$\mathcal{E}\left[\frac{1}{\phi(x)}\right] \cdot \psi(x).$$

But since $\frac{1}{\phi(x)} = \infty$ gives the same roots as $\phi(x) = 0$, which, therefore, is the equation to which we look, we shall find it convenient to use the notation

$$\mathcal{E}\,\frac{\psi(x)}{[\phi(x)]}.$$

Hence, if a be a root of $f(x) = \infty$, the residue with regard to it may be expressed by $\mathcal{E}\,\frac{(x-a)f(x)}{[x-a]}$ instead of $\mathcal{E}_a f(x)$, so that

$$\mathcal{E}\,\frac{(x-a)f(x)}{[x-a]} = h\,f(a+h) \text{ when } h = 0,$$

if there be only one root of $f(x) = \infty$; and

$$\mathcal{E}\,\frac{(x-a)f(x)}{[x-a]} = \frac{1}{1.2\ldots(m-1)}\,\frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0,$$

if there be m roots equal to a.

Now suppose $f(x)$ to be of the form $\frac{\mathrm{f}(x)}{F(x)}$, and that a is a root of $F(x) = 0$; then, as $F(a) = 0$; we have

$$F(a+h) = h\,F'(a+\theta h),$$

and therefore [148]

$$h\,f(a+h) = \frac{h\,\mathrm{f}(a+h)}{h\,F'(a+\theta h)}.$$

So that when $h = 0$,

$$\mathcal{E}_a\,\frac{\mathrm{f}(x)}{[F(x)]} = \frac{\mathrm{f}(a)}{F'(a)} \;\ldots\ldots\ldots\ldots\; (5).$$

As an example, take $\frac{x^m}{x-a}$, whence

$$\mathcal{E}_a\,\frac{x^m}{[x-a]} = a^m.$$

Also, $$\mathcal{E}_a\,\frac{x^2 - px + q}{[x^2 - a^2]} = \frac{a^2 - pa + q}{2a},$$

$$\mathcal{E}_a\,\frac{x^m}{[(x-a)(x-b)]} = \frac{a^m}{a-b}.$$

M

If the fraction be of the form $\dfrac{\mathrm{f}(x)}{(x-a)(x-b)\ldots(x-r)}$, where there are r factors in the denominator, we may extract the residue with respect to each factor, and the sum of all these partial residues will be the integral residue with respect to the denominator. Hence

$$\mathcal{E}\frac{\mathrm{f}(x)}{[(x-a)(x-b)\ldots(x-r)]} = \frac{\mathrm{f}(a)}{(a-b)(a-c)\ldots(a-r)}$$
$$+\frac{\mathrm{f}(b)}{(b-a)(b-c)\ldots(b-r)} + \text{\&c.} + \frac{\mathrm{f}(r)}{(r-a)(r-b)\ldots(r-q)}.$$

When the denominator has m roots, each equal to a,

$$\mathcal{E}_a f(x) = \mathcal{E}_a\frac{\mathrm{f}(x)}{[F(x)]} = \frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\}$$

when $h = 0$.

But $$f(a+h) = \frac{\mathrm{f}(a+h)}{F(a+h)}, \text{ and}$$

$$F(a+h) = F(a) + F'(a)\frac{h}{1} + \text{\&c.} + \frac{F^{(m-1)}(a)\,h^{m-1}}{1.2\ldots(m-1)} + \frac{F^{(m)}(a)\,h^m}{1.2\ldots m} + \text{\&c.}$$

and as these are m equal roots of $F(x) = 0$, we have

$$F(a) = 0,\quad F'(a) = 0\ \ldots.\ F^{(m-1)}(a) = 0.$$

Hence $$F(a+h) = \frac{h^m}{1.2\ldots m}\left\{F^{(m)}(a) + F^{(m+1)}(a)\frac{h}{m+1} + \text{\&c.}\right\},$$

or $$F(a+h) = \int^m dh^m \frac{d^m}{da^m} F(a+h);^*$$

[149] whence we find

$$\mathcal{E}_a\frac{\mathrm{f}(x)}{[F(x)]} = \frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}}{dh^{m-1}}\left\{\frac{h^m\mathrm{f}(a+h)}{\int^m dh^m \frac{d^m}{da^m}F(a+h)}\right\}$$

when $h = 0$ . . . . (6).

When $F(a+h)$ is of the form $(x-a)^m$, this expression is considerably simplified. For

$$\frac{d^m}{dx^m}(x-a)^m = m(m-1)\ldots 2.1,$$

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* By the notation $\dfrac{d^m}{da^m}f(a+h)$, I mean to express the value which $\dfrac{d^m}{dx^m}f(x+h)$ takes when $x = a$.

and $$\int^m dh^m \frac{d^m}{da^m} F(a+h) = h^m; \text{ so that}$$

$$\mathcal{E}_a \frac{\mathrm{f}(x)}{(x-a)^m} = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{\mathrm{f}(a+h)\} \text{ when } h = 0 \ldots (7).$$

Thus $$\mathcal{E}_a \frac{x^n}{(x-a)^4} = \frac{n.(n-1)(n-2)}{1.2.3} a^{n-3}.$$

I shall now proceed to the proof of one of the fundamental theorems of the Residual Calculus. It is, that

$$f(x) - \mathcal{E} \frac{f(z)}{x-z} = \psi(x) \ldots\ldots\ldots\ldots (8),$$

$\psi(x)$ being a function of (x), which takes generally a finite value when $x = a_1, = a_2, = a_3$, &c.,

a_1, a_2, a_3, being the roots of the equation $f(x) = \infty$.

We have generally

$$f(x) = \frac{R(x)}{(x-a_1)^m} \text{ when } x = a_1.$$

Expanding $R(x) = R(a_1 + h)$ by Taylor's theorem, and putting $x - a_1$ for h,

$$f(x) = \frac{R(a_1)}{(x-a_1)^m} + \frac{1}{1} \frac{R'(a_1)}{(x-a_1)^{m-1}} + \&c.$$

$$+ \frac{1}{1.2\ldots(m-1)} \frac{R^{(m-1)}(a_1)}{x-a_1} + \psi(x) \ldots\ldots (9),$$

where $\psi(x)$ is a function of x, which generally takes for $x = a_1$ the finite value $\dfrac{R^{(m)}(a_1)}{1.2\ldots m}$.

But we have also from the definition of a residue

$$\frac{R^{(m)}(a_1)}{1.2\ldots m} = \mathcal{E}_{a_1} \frac{R(z)}{(z-a_1^m)^{+1}} \ldots\ldots\ldots\ldots (10),$$

and similarly for the others. Hence [150]

$$f(x) = \frac{1}{(x-a_1)^m} \{\mathcal{E}_{a_1} \frac{R(z)}{z-a_1} + \mathcal{E}_{a_1} \frac{R(z)}{(z-a_1)^2}(x-a_1)$$

$$+ \mathcal{E}_{a_1} \frac{R(z)}{(z-a_1)^3}(x-a)^2 + \&c.\}$$

$$= \mathcal{E}_{a_1} \frac{R(z)}{(x-a_1)^m (z-a_1)^m} \{(z-a_1)^{m-1} + (z-a_1)^{m-2}(x-a_1) + \&c.\} + \psi(x)$$

$$= \mathcal{E}_{a_1} \frac{R(z)}{(x-a_1)^m (z-a_1)^m} \left\{\frac{(x-a_1)^m - (z-a_1)^m}{x-z}\right\} + \psi(x)$$

$$= \mathcal{E}_{a_1} \frac{R(z)}{(x-z)(z-a_1)^m} - \frac{1}{(x-a_1)^m} \mathcal{E}_{a_1} \frac{R(z)}{x-z} + \psi(x).$$

Now the second term of this expression is equal to 0, because $\frac{R(z)}{x-z}$ does not become infinite for $z = a_1$, and consequently has no corresponding residue. We have therefore

$$f(x) = \mathcal{E}_{a_1} \frac{R(z)}{(x-z)(z-a_1)^m} + \psi(x) = \mathcal{E}_{a_1} \frac{f(z)}{x-z} + \psi(x),$$

since $R(z) = (z - a_1)^m f(z)$.

Hence, in order to deduce from $f(x)$, which becomes infinite for $x = a_1$, a function of x, which shall remain finite under the same circumstances, we must subtract from $f(x)$ the partial residue with respect to a_1 of $\frac{f(z)}{x-z}$. Similarly, if we wish to find a function of x which shall not become infinite when $x = a_1, = a_2$, &c. we must subtract from $f(x)$ the partial residues with respect to each of these roots. So that if we make

$$f(x) - \mathcal{E}\frac{[f(z)]}{x-z} = \phi(x),$$

$\phi(x)$ will have a finite value when $x = a_1, = a_2, = a_3$, &c.

The notation $\mathcal{E}\frac{[f(z)]}{x-z}$, it must be remembered, means the sum of all the residues with respect to the roots of the equation $f(z) = \infty$.

From this theorem we can deduce a number of important results. If $f(x)$ is a fraction, as $\frac{\mathrm{f}(x)}{F(x)}$, $\phi(x)$ must be a fraction of the same kind, the denominator of which must never become 0, since the fraction $\phi(x)$ can never become infinite.

Therefore the denominator must be independent of x or constant, and $\phi(x)$ an integral function of x. If $F(x)$ be of higher dimensions than $\mathrm{f}(x)$, $f(x)$ will vanish for $x = \infty$, as likewise will the residue, so that we must also have $\phi(x) = 0$. Hence in this case

$$f(x) = \mathcal{E}\frac{[f(z)]}{x-z} \quad \ldots\ldots\ldots\ldots\ldots (11),$$

[151] which gives the means of decomposing a rational fraction unto simple fractions.

As an example, take the fraction $\frac{x^2 - 7x + 1}{(x-1)(x-2)(x-3)} = f(x)$.

$$f(x) = \mathcal{E}\,\frac{z^2 - 7z + 1}{(x - z)\,[(z - 1)\ (z - 2)\,(z - 3)]} =$$

$$= \mathcal{E}_1 \frac{z^2 - 7z + 1}{(x - z)(z - 1)(z - 2)(z - 3)} + \mathcal{E}_2 \frac{z^2 - 7z + 1}{(x - z)(z - 1)(z - 2)(z - 3)}$$

$$+ \mathcal{E}_3 \frac{z^2 - 7z + 1}{(x - z)\,(z - 1)\,(z - 2)\,(z - 3)}$$

$$= -\frac{5}{2}\,\frac{1}{x - 1} + 9\,\frac{1}{x - 2} - \frac{11}{2}\,\frac{1}{x - 3}.$$

Again, take $$\frac{1}{(x - 1)^2\,(x + 1)} = f(x)$$

$$f(x) = \mathcal{E}_{-1}\,\frac{1}{(x - z)(z - 1)^2\,(z + 1)} + \mathcal{E}_1 \frac{1}{(x - z)\,(z - 1)^2\,(z + 1)}$$

$$= \tfrac{1}{4}\,\frac{1}{x + 1} + \frac{d}{dh}\,\frac{1}{(x - 1 - h)\,(2 + h)} \text{ when } h = 0$$

$$= \tfrac{1}{4}\,\frac{1}{x + 1} + \tfrac{1}{2}\,\frac{1}{(x - 1)^2} - \tfrac{1}{4}\,\frac{1}{x - 1}.$$

When the fraction is of the form $\frac{\mathrm{f}(x)}{(x - a)^m}$, we can easily deduce by this method the series of component simple fractions.

For $$\frac{\mathrm{f}(x)}{(x - a)^m} = \mathcal{E}_a\,\frac{\mathrm{f}(z)}{(x - z)\,(z - a)^m}$$

$$= \frac{1}{1.2\ldots(m - 1)}\,\frac{d^{m-1}}{dh^{m-1}}\left(\frac{\mathrm{f}(a + h)}{x - a - h}\right) \text{ when } h = 0\,;$$

and if we actually perform the operations indicated, we get

$$\frac{\mathrm{f}(x)}{(x - a)^m} = \frac{1}{1.2\ldots(m - 1)}\left\{\frac{\mathrm{f}^{(m-1)}(a)}{(x - a)} + (m - 1)\,\frac{\mathrm{f}^{(m-2)}(a)}{(x - a)^2}\right.$$

$$+ (m - 1)\,(m - 2)\,\frac{\mathrm{f}^{(m-3)}(a)}{(x - a)^3} + \&\text{c.}$$

$$\left. + (m - 1)\,(m - 2)\ldots 2.1\,\frac{\mathrm{f}(a)}{(x - a)^m}\right\} \ldots\ldots (12).$$

This also appears from (9) by inverting the order of the terms.

By the application of the theorem (11) we can easily prove Lagrange's formula for interpolation. For if

$$f(x) = \frac{\mathrm{f}(x)}{(x - x_1)\,(x - x_2)\ldots(x - x^m)},$$

[152] we have

$$\mathcal{E}\,\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\,\frac{1}{x-z}$$

$$=\mathcal{E}_{x_1}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\,\frac{1}{x-z}$$

$$+\,\mathcal{E}_{x_2}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\,\frac{1}{x-z}+\&c.$$

$$+\,\mathcal{E}_{x_m}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\,\frac{1}{x-z}$$

$$=\frac{\mathrm{f}(x_1)}{(x_1-x_2)\ldots(x_1-x_m)}\,\frac{1}{x-x_1}+\&c.$$

$$+\,\frac{\mathrm{f}(x_m)}{(x_m-x_1)\ldots x_m-x_{m-1}}\,\frac{1}{x-x_m};$$

whence $\mathrm{f}(x)=\dfrac{(x-x_2)\ldots(x-x_m)}{(x_1-x_2)\ldots(x_1-x_m)}\,\mathrm{f}(x_1)+\&c.$

$$+\,\frac{(x-x_1)\ldots x-x_{m-1}}{(x_m-x_1)\ldots x_m-x_{m-1}}\,\mathrm{f}(x_m).$$

Again, taking the equation

$$\mathrm{f}(x)=\mathcal{E}\,\frac{[f(z)]}{x-z},$$

and multiplying both sides by x, it becomes

$$xf(x)=\mathcal{E}\,\frac{[f(z)]}{1-\dfrac{z}{x}}.$$

On making $x=\infty$, $f(x)=0$, as the denominator of $f(x)$ is supposed to be of higher dimensions than the numerator; therefore $xf(x)$ may have a finite value. Let this be V, then

$$\mathcal{E}f(z)=V \;\ldots\ldots\ldots\ldots\ldots\; (13).$$

If $V=0$, which will be the case when the degree of the denominator of $f(x)$ surpasses that of the numerator by more than unity, then we have simply

$$\mathcal{E}f(z)=0 \;\ldots\ldots\ldots\ldots\ldots\; (14).$$

Let $\qquad f(x)=\dfrac{x^n}{(x-x_1)(x-x_2)\ldots(x-x_m)}$,

we find that

$$\frac{x_1^{\,n}}{(x_1-x_2)\ldots(x_1-x_m)}+\frac{x_2^{\,n}}{(x_2-x_1)\ldots(x_2-x_m)}+\&c.$$

$$+\,\frac{x_m^{\,n}}{(x_m-x_1)\ldots(x_m-x_{m-1})}=1 \quad\text{or}\quad =0,$$

according as $n = m - 1$, or as $n < m - 1$; since in the former case we have $xf(x) = 1$ when $x = \infty$, and in the latter [153] $xf(x) = 0$ when $x = \infty$.

We have as yet limited ourselves to the consideration of proper fractions, but the Residual Calculus may be extended to all others. To do this, we must first prove the following theorem,

$$\mathcal{E} f(z) = \mathcal{E} \frac{f\left(\dfrac{1}{z}\right)}{[z^2]} \ldots\ldots\ldots\ldots (15).$$

We know by (8) that

$$\frac{f\left(\dfrac{1}{u}\right)}{u^2} = \mathcal{E} \frac{f\left(\dfrac{1}{s}\right)}{[s^2](u-s)} + U,$$

where U is a function of u, which remains finite for $u = 0$.

Let there be m roots of $f\left(\dfrac{1}{s}\right) = \infty$, each equal to 0, and let us expand $\dfrac{1}{u-s}$; then

$$\frac{f\left(\dfrac{1}{u}\right)}{u^2} = \frac{1}{u}\mathcal{E}\frac{f\left(\dfrac{1}{s}\right)}{[s^2]} + \frac{1}{u^2}\mathcal{E}\frac{sf\left(\dfrac{1}{s}\right)}{[s^2]} + \text{\&c.} + \frac{1}{u^{m+2}}\mathcal{E}\frac{s^{m+1}f\left(\dfrac{1}{s}\right)}{[s^2]} + U.$$

Multiplying by u^2, substituting z for $\dfrac{1}{u}$, and making $Uu^2 = \phi(z)$,

$$f(z) = \frac{1}{z}\mathcal{E}\frac{f\left(\dfrac{1}{s}\right)}{[s^2]} + \mathcal{E}\frac{sf\left(\dfrac{1}{s}\right)}{[s^2]} + \text{\&c.} + z^m\,\mathcal{E}\frac{s^{m+1}f\left(\dfrac{1}{s}\right)}{[s^2]} + \phi(z).$$

Now, extracting the residues with regard to z, and observing that

$$\mathcal{E}\left(\frac{1}{z}\right) = 1, \quad \mathcal{E}\,1 = 0, \quad \mathcal{E}\,z = 0 \ldots \mathcal{E}\,z^m = 0,$$

we find
$$\mathcal{E} f(z) = \mathcal{E}\frac{f\left(\dfrac{1}{s}\right)}{[s^2]} + \mathcal{E}\,\phi(z) \ldots\ldots\ldots (16).$$

Now $\phi(z) = Uu^2$, and U has a finite value for $u = 0$, therefore $Uu = z\phi(z)$ must vanish. Hence by (14)

$$\mathcal{E}\,\phi(z) = z\,\phi(z) = 0.$$

Hence, finally, we have

$$\mathcal{E} f(z) = \mathcal{E}\frac{f\left(\dfrac{1}{s}\right)}{[s^2]} = \mathcal{E}\frac{f\left(\dfrac{1}{z}\right)}{[z^2]} \ldots\ldots (17).$$

[154] In this last formula substitute $\frac{f(z)}{z-x}$ for $f(z)$. As the residue is taken with regard to all the roots of $\frac{f(z)}{z-x}=\infty$, and one of these is $x=z$, we may separate that partial residue from the others. Thus we have

$$\mathcal{E}\frac{f(z)}{z-x}=\mathcal{E}_x\frac{f(z)}{z-x}+\mathcal{E}\frac{[f(z)]}{z-x}$$

$$=f(x)+\mathcal{E}\frac{[f(z)]}{z-x};$$

so that, by the last theorem,

$$f(x)+\mathcal{E}\frac{[f(z)]}{z-x}=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]\left(\frac{1}{z}-x\right)}=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}$$

$$\text{or}\quad f(x)=\mathcal{E}\frac{[f(z)]}{x-z}+\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}\ \ldots\ (18),$$

which is the equation we ought to substitute for the formula (11), if $f(z)$ becomes infinite when $z=\infty$.

If $f(x)$ is an integral function of x, the residue $\mathcal{E}\frac{[f(z)]}{x-z}$ vanishes, since $f(z)$ has a finite value for any value of z, so that the expression is reduced to

$$f(x)=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}\ \ldots\ldots\ldots\ (19).$$

If $f(x)$ be of the form $\frac{\mathrm{f}(x)}{F(x)}$, and $F(x)$ be of higher dimensions than $\mathrm{f}(x)$, the expression (18) is, as was proved before, reduced to $\mathcal{E}\frac{[f(z)]}{x-z}$, which enables us to decompose a rational fraction. But if $F(x)$ be of lower dimensions than $\mathrm{f}(x)$, the function $\frac{\mathrm{f}(x)}{F(x)}$ is, by (18), divided into two parts, of which $\mathcal{E}\frac{[f(z)]}{x-z}$ is the sum of the rational fractions remaining from the division of $\mathrm{f}(x)$ by $F(x)$, and $\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}$ is the quotient arising from that division.

Let us take, as an example, $f(x) = \dfrac{1 + x^4}{x(1 + x^2)}$ [155]

$$f(x) = \mathcal{E}\frac{(1 + z^4)}{[z(1 + z^2)](x - z)} + \mathcal{E}\frac{1 + z^4}{[z^2](1 + z^2)(1 - zx)}.$$

Now

$$\mathcal{E}\frac{1 + z^4}{[z(1+z^2)](x-z)} = \mathcal{E}_0\frac{(1 + z^4)}{z(1+z^2)(x-z)} + \mathcal{E}_{+\sqrt{-1}}\frac{1 + z^4}{z(1+z^2)(x-z)}$$

$$+ \mathcal{E}_{-\sqrt{-1}}\frac{1 + z^4}{z(1 + z^2)(x - z)}$$

$$= \frac{1}{x} - \frac{1}{x + \sqrt{-1}} - \frac{1}{x - \sqrt{-1}},$$

and $\mathcal{E}\dfrac{1 + z^4}{[z^2](1 + z^2)(1 - zx)} = \dfrac{d}{dh}\dfrac{(1 + h^4)}{(1 + h^2)(1 - xh)}$ when $h = 0, = x$.

Therefore, finally,

$$\frac{1 + x^4}{x(1 + x^2)} = x + \frac{1}{x} - \frac{1}{x + \sqrt{-1}} - \frac{1}{x - \sqrt{-1}}$$

$$= x + \frac{1}{x} - \frac{2x}{1 + x^2}.$$

The length to which this article has extended, renders it necessary to postpone to a future Number other illustrations of the use of the Residual Calculus.

D. F. G.

TRANSFORMATION OF CERTAIN ANALYTICAL EXPRESSIONS.

We frequently require in Analysis to find the sum of the squares of three quantities of the form

$$ay - bx, \quad bz - cy, \quad cx - az,$$

and a slight artifice enables us to do this very readily.

By adding and subtracting $a^2x^2 + b^2y^2 + c^2z^2$, the sum of the squares becomes

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \ldots\ldots (1)$$

$$= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\left\{1 - \frac{(ax + by + cz)^2}{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)}\right\}.$$

Now the expression $\dfrac{ax + by + cz}{(a^2 + b^2 + c^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}}$ is the cosine of the angle between two lines which make angles with the

[156] coordinate axes, whose cosines are proportional to $a\ b\ c$, $x\ y\ z$ respectively. If this angle be called θ, the expression becomes

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)\sin^2\theta \ldots\ldots\ldots (3).$$

We shall give some examples of the use of this formula.

1. In transforming expressions from one set of rectangular coordinates to another, making with the original axes angles whose cosines are $a\ a'\ a''$, $b\ b'\ b''$, $c\ c'\ c''$, we obtain the relations

$$a = \frac{b'c'' - c'b''}{D}, \quad a' = \frac{cb'' - bc''}{D}, \quad a'' = \frac{bc' - cb'}{D},$$

and we wish to discover the value of D. Squaring these equations, adding and observing that $a^2 + a'^2 + a''^2 = 1$, we find

$$D^2 = (b'c'' - c'b'')^2 + (cb'' - bc'')^2 + (bc' - cb')^2.$$

By formula (3) this becomes

$$D^2 = (b^2 + b'^2 + b''^2)(c^2 + c'^2 + c''^2)\sin^2\theta.$$

Now $\sin\theta = 1$, since the lines are at right angles to each other, the new coordinates being rectangular; also

$$b^2 + b'^2 + b''^2 = 1, \quad \text{and} \quad c^2 + c'^2 + c''^2 = 1,$$

so that $D = 1$, and

$$a = b'c'' - c'b'', \quad a' = cb'' - bc'', \quad a'' = bc' - cb'.$$

2. Let a material point be acted on by any forces whose resolved parts parallel to rectangular axes are X, Y, Z; let ρ be the radius of curvature of its path at any point xyz, then we know that

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^3 =$$

$$\left\{\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right)^2 + \left(\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right)^2 + \left(\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left(Y\frac{dx}{dt} - X\frac{dy}{dt}\right)^2 + \left(Z\frac{dy}{dt} - Y\frac{dz}{dt}\right)^2 + \left(X\frac{dz}{dt} - Z\frac{dx}{dt}\right)^2\right\}^{\frac{1}{2}}.$$

Therefore, by formula (3),

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^3 = \sqrt{(X^2 + Y^2 + Z^2)}\,\frac{ds}{dt}\,.\,\sin\,(\text{angle between tangent and resultant}),$$

or if R be the resultant, V the velocity,

$$v^2 = \rho R \cos\,(\text{angle between resultant and rad. of curvature}).$$

Calling this angle ϕ, $R\cos\phi$ is the part of the resultant resolved perpendicularly to the path, which is opposed and

equal to the centrifugal, which is therefore equal to $\dfrac{v^2}{\rho}$.

$\rho \cos \phi$ is the $\frac{1}{2}$ chord of curvature lying in the direction of the resultant, and therefore

$$v^2 = 2R \tfrac{1}{4} \text{ chord of curvature along resultant,}$$

or the velocity in any curve at any point is equal to that acquired by falling through $\frac{1}{4}$ of the chord of curvature under the influence of the resultant force at the point, considered constant.

3. If p, q, r be the angular velocities of a body [157]
round the coordinate axes, the velocities of a point (xyz) parallel to the axes are

$$qz - ry, \quad rx - pz, \quad py - qx.$$

A point in the line whose equations are

$$\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$$

has no velocity. This line therefore is the axis of instantaneous rotation, and p, q, r are respectively proportional to the cosines of the angles it makes with the coordinate axes.

The velocity of any other point is the square root of the sum of the squares of the velocities parallel to the axes. If this be v we have, by (3),

$$v = \sqrt{(p^2 + q^2 + r^2)}\ \sqrt{(x^2 + y^2 + z^2)} \,.\, \sin\,(\text{angle between inst. axis and line through } xyz)$$

$$= \sqrt{(p^2 + q^2 + r^2)} \,.\, \text{perpendicular on inst. axis.}$$

And $\sqrt{(p^2 + q^2 + r^2)}$, being equal to the linear velocity divided by the distance of the point from the instantaneous axis, is the angular velocity about the axis.

4. The expression for the radius of curvature of a curve of double curvature is

$$\rho = \frac{ds^2\,\{(Ady - Bdx)^2 + (Cdx - Adz)^2 + (Bdz - Cdy)^2\}^{\frac{1}{2}}}{A^2 + B^2 + C^2}$$

where

$$A = dy\,d^2z - dz\,d^2y, \quad B = dz\,d^2x - dx\,d^2z, \quad C = dx\,d^2y - dy\,d^2x,$$

which are proportional to the cosines of the angles which a normal to the osculating plane makes with the coordinate axes. The numerator is, by (3), equal to

$$ds^2\,(A^2 + B^2 + C^2)^{\frac{1}{2}}\,(dx^2 + dy^2 + dz^2)^{\frac{1}{2}} \sin^2 \theta,$$

when θ is the angle between a normal to the osculating

plane and the tangent, and therefore is a right angle, so that $\sin\theta = 1$; and, as $dx^2 + dy^2 + dz^2 = ds^2$, we have

$$\rho = \frac{ds^3}{(A^2 + B^2 + C^2)^{\frac{1}{2}}}.$$

A. S.

DEMONSTRATIONS OF SOME PROPERTIES OF A TRIANGLE.

The following properties of a triangle are not perhaps generally known. If we join D, E, F, the points in which perpendiculars on the sides from the opposite angles intersect [158] the sides, the triangles DEC, DBF, AFE are similar to each other, and to the triangle ABC; and the triangle DEF is the triangle of least perimeter which can be inscribed in the triangle ABC.

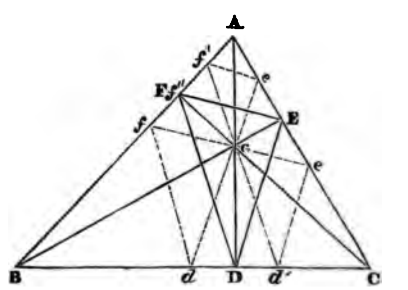
We shall first prove that two adjacent sides of the triangle DEF make equal angles with the side of the triangle ABC; that is, that $BDF = EDC$, $DFB = EFA$, $AEF = DEC$.

We have $$BDF = DFC + DCF,$$
as it is the exterior and opposite angle of the triangle DFC. For a similar reason $EDC = DBE + DEB$.

Now, since the three perpendiculars AD, BE, CF pass through the same point G, and in the quadrilateral $BFGD$ the angles at D and F are together equal to two right angles, the four points B, F, G, D are in the circumference of a circle: and the angles DBG and DFG, or DBE and DFC, being angles in the same segment, are equal to one another. In the same way we see that $DCF = DEB$, consequently $BDF = CDE$, and in a similar manner we might prove that $BFD = EFA$ and $FEA = DEC$.

We shall next prove that $EDC = BAC$, $DEC = ABC$, and $BFD = BCA$. For we have, as before,

$$EDC = DBE + DEB.$$



Now, since ADB and AEB are right angles, they are angles in a semicircle, so that A, B, D, E are points in the circumference of a circle, and DBE and DAE being angles in the same segment of a circle are equal, and also DAB and DEB for the

same reason. But $BAD + DAE = BAE$, so that $EDC = BAC$; and similarly for the others. Comparing the triangles DEC and ABC, we see that the angle at C is common, the angles $CDE = BAC$ and the angle $DEC = ABC$, therefore they are equiangular and similar. In the same way DBF and FAE are similar to ABC, and the three triangles DEC, DBF, and FAE being similar to ABC are similar to one another.

To prove that DEF is the triangle of least perimeter which can be inscribed in ABC, we must avail ourselves of a known property, that if from two points without a line we draw two straight lines to a point in the line, their sum will be a minimum when the two lines make equal angles with the given line. Now, in this case, having proved that FD and ED make equal angles with BC, we see that, supposing F and E fixed, the sum of FD and DE is the least of all those which can be drawn from F and E to any point in BC. The same holds true of the other points F and E, so that on the whole the perimeter FED is the least of all those which can be inscribed in ABC.

Various other properties may be demonstrated of this triangle of least perimeter. If we call the angle $BAD = \alpha$, $EBC = \beta$, $FCA = \gamma$, it is easy to see that $FED = 2\alpha$, $EFD = 2\beta$, $FDE = 2\gamma$, and therefore $2(\alpha + \beta + \gamma)$ is equal to two right angles, or $\alpha + \beta + \gamma$ is equal to a right angle. From the construction of the figure it appears that $\alpha = \frac{\pi}{2} - B$, $\beta = \frac{\pi}{2} - C$, $\gamma = \frac{\pi}{2} - A$. Also, since the angles at D, E, F are bisected by the lines intersecting at G, that point [159] is the centre of the circle inscribed in the triangle of least perimeter. If we draw perpendiculars from the vertices of the triangle ABC to the sides of DEF, it will be seen from the similarity of the triangles that they will divide the angles into the same parts as the perpendiculars on the sides of ABC, but in an inverted position. Now, calling the angles as before α, β, γ, AD makes angles α and β with AF, AE respectively, and therefore the perpendicular on FE makes angles α and β with AE, AF respectively. The position of AD may be determined by expressing the relation between α and β, which may be put under the form $\beta = \phi(\alpha)$. Similarly, the position of BE will be determined by $\gamma = \phi_1(\beta)$, and the position of CF by $\alpha = \phi_2(\gamma)$. But since these lines intersect in one point, one of the three equations must be deducible from the other two, as the combination of two at their point of intersection must

coincide with each other. Now the perpendiculars on the sides of the triangle of least perimeter being defined by the same equations, of which one is derivable from the other two, must all pass through one point. This may be clearer if we take the actual relations, which will be found to be

$$\alpha = C - B + \beta$$
$$\beta = A - C + \gamma$$
$$\gamma = B - A + \alpha,$$

the latter of which is clearly derivable from the other two, so that they are not independent, and therefore the lines expressed by them must all pass through one point.

The perimeter of the triangle DEF may be easily found. For, from the similarity of the triangles ABC, DEF, we have

$$FD : BD = AC :: AB = b : c, \quad \text{or} \quad FD = \frac{b}{c} BD,$$

but $\qquad BD = c \cos B$, so that $FD = b \cos B$.

Similarly, $DE = c \cos C$, and $FE = a \cos A$; so that if p be the perimeter of DEF,

$$p = a \cos A + b \cos B + c \cos C.$$

We may likewise obtain an expression for the perimeter involving only the sides of the triangle ABC. The area of the triangle DEF

$$= \tfrac{1}{2} FD \,.\, DE \sin 2\gamma = \tfrac{1}{2} bc \cos B \cos C \sin 2A.$$

Since $\qquad \gamma = \dfrac{\pi}{2} - A$ and $2\gamma = \pi - 2A$.

Now if r be the radius of the circle inscribed in DEF, its area $= \tfrac{1}{2} pr$.

But $\quad r = GD \sin \gamma = GD \cos A = CD \tan \alpha \cos A$

$$= b \cos C \cos A \tan \alpha = b \, \frac{\cos A \cos B \cos C}{\sin B}.$$

[160] Therefore

$$\tfrac{1}{2} bp \, \frac{\cos A \cos B \cos C}{\sin B} = \tfrac{1}{2} bc \cos B \cos C \sin 2A$$
$$= bc \cos A \cos B \cos C \sin A,$$

or $$p = 2c \sin A \sin B.$$

Or, substituting for $\sin A$ and $\sin B$, their values in terms of the sides of ABC,

$$p = 2c \, \frac{2}{bc} \cdot \frac{2}{ac} \left\{ S(S - a)(S - b)(S - c) \right\}$$

or $$p = \frac{8}{abc}\{S(S-a)(S-b)(S-c)\}.$$

If R' be the radius of the circle described round ABC, we have, by taking its value in terms of the sides of the triangle,

$$p = \frac{abc}{2R'^2}.$$

Also, if R be the radius of the circle inscribed in ABC, we know that, generally,

$$2RR' = \frac{abc}{a+b+c} = \frac{abc}{P},$$

if P be the perimeter of ABC. Substituting the value of R'^2, from this we have

$$p = 2\,\frac{P^2R^2}{abc}.$$

And therefore if K be the area of ABC,

$$p = \frac{8K^2}{abc}.$$

The area of DEF is very easily found in terms of the area of ABC, for we found before that it was equal to

$$\tfrac{1}{2}bc \sin 2A \cos B \cos C = bc \sin A \cos A \cos B \cos C$$
$$= 2K \cos A \cos B \cos C = k \text{ suppose.}$$

If a', b', c' be the sides of DEF,

$$a' = a \cos A, \quad b' = b \cos B, \quad c' = c \cos C,$$

and therefore substituting for $\cos A \cos B \cos C$,

$$k = 2K\,\frac{a'b'c'}{abc}.$$

But we have $abc = 2PRR' = 4KR'$,

and therefore $$k = \frac{a'b'c'}{2R}.$$

It will be seen at once that the areas of the triangles FAE, DBF, DCE, are respectively

$$K \cos^2 A, \quad K \cos^2 B, \quad K \cos^2 C:$$

as the sum of these together with DEF make up ABC, [161] we have, dividing by K,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\,\frac{a'b'c'}{abc} = 1.$$

A singular relation exists between the radii of the circles

described round ABC and DEF. Let them be R' and r' respectively. Then

$$a = 2R' \sin A, \; a' = 2r' \sin 2\gamma = 2r' \sin 2A.$$

Therefore $\quad a' = a \cos A = 4r' \sin A \cos A.$

Whence $\quad R' = 2r'.$

We found for r the radius of the circle inscribed in DEF,

$$r = \frac{b}{\sin B} \cos A \cos B \cos C,$$

and as $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R';$$

therefore $$r = 2R' \cos A \cos B \cos C = R' \frac{k}{K}.$$

Also, since $$r = \frac{2k}{p} = \frac{4K}{p} \cos A \cos B \cos C,$$

$$R' = \frac{2K}{p}, \text{ and } r' = \frac{K}{p}.$$

Since the sides of any one of the triangles round DEF are equal to the sides of the original triangle multiplied by the cosine of the corresponding angle, all similarly situate lines in the triangles, and therefore the radii of the circumscribing circles, will have the same ratio.

Now the circles described round FAE passes through G, as A, F, E, G, are in the circumference of the same circle: and as AEG, AFG are right angles, AG must be the diameter of the circle, therefore $AG = 2R' \cos A$.

Similarly, $BG = 2R' \cos B$, $CG = 2R' \cos C$.

We find also,

$$GD = 2R' \cos B \cos C, \; GE = 2R' \cos A \cos C,$$
$$GF = 2R' \cos A \cos B.$$

To these geometrical properties we may add the following optical one.

If the interior sides of the triangle be reflecting surfaces, and a ray of light be incident on one of the sides in a direction parallel to one of the sides of the inscribed triangle as to FD, it will, after two reflexions from each side, return to its original direction so as to continue the same course *ad infinitum*. This property depends of course on the equality of the angles of incidence and reflection, so that the ray being incident parallel to one of the sides of FDE will continue its course successively parallel to the other sides. Let fd be the incident ray which being incident parallel to FD will be

reflected at d parallel to DE, and falling on AC at e is reflected parallel to EF meeting AB in f'. Proceeding from that point in the same manner, it will after reflection again from BC and AC meet AB again in some point f'': our object is to shew that f'' coincides with f. For if it does so, it is clear from the course of the ray being parallel to the sides of EFD that it will go over the same track again, and so on. [162]

Since the triangles Bdf, ABC are similar, we have $BC.Bd = AB.Bf$. For the like reason we have

$$BC.Cd = AC.Ce$$

$$AB.Af' = AC.Ae$$

$$AB.Bf' = BC.Bd'$$

$$AC.Ce' = BC.Cd'$$

$$AC.Ae' = AB.Af''.$$

Adding these together, and observing that we have

$$Bd + Cd = BC, \quad Ce + Ae = AC, \quad Af' + Bf' = AB,$$

$$Ae' + Ce' = AC, \quad Bd' + Cd' = BC,$$

we have

$$BC^2 + AB^2 + AC^2 = AC^2 + BC^2 + AB.Bf + AB.Af'',$$

whence $$AB^2 = AB(Bf + Af''),$$

or $$AB = Bf + Af''.$$

Therefore $Af = Af''$ and f' and f'' coincide, and the proposition is manifest.

This and several of the other properties of the triangle proved here are due to Professor Wallace of Edinburgh.

V.

ON THE SOLUTION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS.*

INTEGRATION of the Partial Differential Equation

$$X^a Y^b Z^c \left(\frac{dZ}{dX}\right)^m \left(\frac{dZ}{dY}\right)^n - A = 0,$$

a, b, c, m, n being any quantities whatever.

* From a Correspondent.

This equation may be put under the form

$$\left\{X^{\frac{a}{m}}\, Z^{\frac{c}{m+n}}\, \frac{dZ}{dX}\right\}^m . \left\{Y^{\frac{b}{n}}\, Z^{\frac{c}{m+n}}\, \frac{dZ}{dy}\right\}^n - A = 0.$$

[163] Let $x = \int X^{-\frac{a}{m}} dX = \dfrac{m}{m-a}\, X^{\frac{m-a}{m}}$

$$y = \int Y^{-\frac{b}{n}} dY = \frac{n}{n-b}\, Y^{\frac{n-b}{n}}, \quad z = \int Z^{\frac{c}{m+n}} dZ = \frac{m+n}{m+n+c}\, Z^{\frac{m+n+c}{m+n}},$$

and the equation becomes

$$\left(\frac{dz}{dx}\right)^m \left(\frac{dz}{dy}\right)^n - A = 0 \;\ldots\ldots\ldots\ldots (1),$$

which is easily integrable.

If $m = a$, then $x = \log_e X$; if $n = b$, then $y = \log_e Y$; and if $m + n + c = 0$, then $Z = \log_e Z$.

But if $m + n = 0$, and c does not $= 0$, then (and in this case only) the above substitution for Z will not answer, and equation (1) must stand thus,

$$\left(\frac{dZ}{dx}\right)^m \left(\frac{dZ}{dy}\right)^n Z^c - A = 0 \;\ldots\ldots\ldots (2).$$

But by considering x as a function of the independent variables Z and y, we may transform this equation so as to be easily integrated.

For $$dx = \frac{dx}{dZ}\, dZ + \frac{dx}{dy}\, dy;$$

$$\therefore\; dZ = \frac{1}{\frac{dZ}{dx}}\, dx - \frac{\frac{dx}{dy}}{\frac{dx}{dZ}}\, dy;$$

and hence $\dfrac{dZ}{dx} = 1 \div \dfrac{dx}{dZ}$, and $\dfrac{dZ}{dy} = -\dfrac{dx}{dy} \div \dfrac{dx}{dZ}$;

and equation (2) becomes by substitution (since $m + n = 0$)

$$\left(\frac{dx}{dy}\right)^n Z^c - A\,(-1)^n = 0,$$

the integral of which is

$$x = -\,y Z^{-\frac{c}{n}} A^{\frac{1}{n}} + F(Z).$$

The same process may be applied when there is any number of independent variables.

Ex. 1. $\dfrac{dz}{dy} = \dfrac{dz^2}{dx^2}\, xy^2 z^3.$

This may be written $\dfrac{z^3}{y^2}\dfrac{dz}{dy} = \left(x^{\frac{1}{2}} z^3 \dfrac{dz}{dx}\right)^2$:

put $\dfrac{y^3}{3} = y'$, $2x^{\frac{1}{2}} = x'$, and $\dfrac{z^4}{4} = z'$, and we have

$$\frac{dz'}{dy'} = \frac{dz'^2}{dx'^2}.$$

To integrate this we proceed as follows: [164]

$$dz' = \frac{dz'}{dx'}\,dx' + \frac{dz'}{dy'}\,dy' = p\,dx' + q\,dy' \text{ suppose;}$$

$$\therefore z' = px' + qy' - \int(x'\,dp + y'\,dq) = px' + p^2y' - \int(x' + 2py')\,dp;$$

$$\left.\begin{aligned} \therefore\ x' + 2py' &= F'(p), \\ \text{and}\qquad z' &= px' + p^2y' - F(p), \end{aligned}\right\}$$

from which p must be eliminated.

If we restore the original variables, these equations are

$$\left.\begin{aligned} 2\sqrt{x} + \tfrac{2}{3}y^3p &= F'(p), \\ \tfrac{1}{4}z^4 &= 2\sqrt{x}\,p + \tfrac{1}{3}y^3p^2 - F(p). \end{aligned}\right\}$$

Ex. 2. $\dfrac{dz}{dy}\,xz = \dfrac{dz}{dx}$:

this comes under the case where $m+n=0$, and c does not $=0$.

Put $\dfrac{x^2}{2} = x'$, and the equation becomes

$$\frac{dz}{dy}\,z = \frac{dz}{dx'},$$

$$dx' = \frac{dx'}{dz}\,dz + \frac{dx'}{dy}\,dy;$$

$$\therefore\ dz = \left(1 \div \frac{dx'}{dz}\right)dx' - \left(\frac{dx'}{dy} \div \frac{dx'}{dz}\right)dy;$$

$$\therefore\ \frac{dz}{dx'} = 1 \div \frac{dx'}{dz}, \text{ and } \frac{dz}{dy} = -\frac{dx'}{dy} \div \frac{dx'}{dz},$$

and our equation becomes

$$\frac{dx'}{dy}\,z = -1;$$

$$\therefore x' = -\frac{y}{z} + F(z),$$

or
$$\frac{x^2}{2} + \frac{y}{z} = F(z).$$

G. C.

[165]

ON THE VARIATION OF THE LONGITUDE OF PERIHELION IN THE PLANETARY THEORY.

THE following is a short method of finding the expression for $\frac{d\omega}{dt}$.

The usual notation is employed.

We have the equation

$$\frac{1}{r} = \frac{\mu}{h^2}\{1 + e\cos(\theta - \omega)\} \dots\dots\dots\dots (1);$$

and since
$$1 - e^2 = \frac{h^2}{\mu a},$$

e is a function of a and h, therefore

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} + \frac{dr}{da}\frac{da}{dt} + \frac{dr}{dh}\frac{dh}{dt} + \frac{dr}{d\omega}\frac{d\omega}{dt};$$

but
$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt};$$

therefore, observing that

$$\frac{dr}{d\omega} = -\frac{dr}{d\theta},$$

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = \frac{dr}{da}\frac{da}{dt} + \frac{dr}{dh}\frac{dh}{dt}.$$

Now $\frac{da}{dt} = -\frac{2a^2}{\mu}\frac{d(R)}{dt}$, and $\frac{dh}{dt} = -\frac{dR}{d\theta}$; therefore

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = -\frac{2a^2}{\mu}\frac{dr}{da}\frac{d(R)}{dt} - \frac{dr}{dh}\frac{dR}{d\theta},$$

and
$$\frac{d(R)}{dt} = \frac{dR}{dr}\frac{dr}{dt} + \frac{dR}{d\theta}\frac{d\theta}{dt};$$

therefore

$$\frac{dr}{d\theta}\frac{d\varpi}{dt} = -\frac{2a^2}{\mu}\frac{dr}{da}\frac{dr}{dt}\frac{dR}{dr} - \left(\frac{2a^2}{\mu}\frac{dr}{da}\frac{d\theta}{dt} + \frac{dr}{dh}\right)\frac{dR}{d\theta}$$

$$= -\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da}\frac{dr}{d\theta}\frac{dR}{dr} - \left(\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da} + \frac{dr}{dh}\right)\frac{dR}{d\theta} \dots (2),$$

putting $\qquad \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt}$, and $\dfrac{d\theta}{dt} = \dfrac{h}{r^2}$.

Now, differentiating (1) relatively to a, we have

$$-\frac{1}{r^2}\frac{dr}{da} = \frac{\mu}{h^2}\frac{de}{da}\cos(\theta - \varpi) \dots\dots\dots\dots (3):$$

and since $1 - e^2 = \dfrac{h^2}{\mu a}$,

$$-2e\frac{de}{da} = -\frac{h^2}{\mu a^2}, \text{ and } -2e\frac{de}{dh} = \frac{2h}{\mu a},$$

whence $\qquad \dfrac{de}{da} = -\dfrac{h}{2a}\dfrac{de}{dh}$, and (3) becomes [166]

$$-\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da} = -a\frac{de}{dh}\cos(\theta - \varpi) \dots\dots\dots (4).$$

Also, $\qquad r = a(1 - e\cos u),$

$$u = nt + \epsilon - \varpi + e\sin u;$$

therefore $\qquad \dfrac{d(r)}{dh} = -a\dfrac{de}{dh}\cos u + ae\sin u\dfrac{du}{dh}$,

and $\qquad \dfrac{du}{dh} = \dfrac{de}{dh}\sin u + e\cos u\dfrac{du}{dh}$,

hence $\qquad \dfrac{du}{dh} = \dfrac{de}{dh}\cdot\dfrac{\sin u}{1 - e\cos u}$,

$$\frac{d(r)}{dh} = -a\frac{de}{dh}\left(\cos u - \frac{e\sin^2 u}{1 - e\cos u}\right)$$

$$= -a\frac{de}{dh}\frac{\cos u - e}{1 - e\cos u}$$

$$= -a\frac{de}{dh}\cos(\theta - \varpi);$$

hence, by (4), $\qquad -\dfrac{2a^2}{\mu}\dfrac{h}{r^2}\dfrac{dr}{da} = \dfrac{d(r)}{dh}$.

Substituting in (2)

$$\frac{dr}{d\theta}\frac{d\varpi}{dt} = \frac{dr}{d\theta}\frac{dR}{dr}\frac{d(r)}{dh} + \frac{dR}{d\theta}\left\{\frac{d(r)}{dh} - \frac{dr}{dh}\right\}:$$

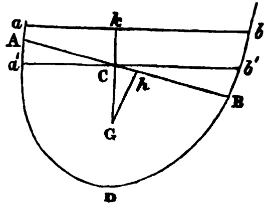
but
$$\frac{d(r)}{dh} = \frac{dr}{dh} + \frac{dr}{d\theta}\frac{d(\theta)}{dh};$$

therefore
$$\frac{d\omega}{dt} = \frac{dR}{dr}\frac{d(r)}{dh} + \frac{dR}{d\theta}\frac{d(\theta)}{dh}$$
$$= \frac{dR}{dh}$$
$$= \frac{dR}{de}\frac{de}{dh}$$
$$= -\frac{na\sqrt{(1-e)^2}}{\mu e}\frac{dR}{de}.$$

M. O. B.

[167] ON THE OSCILLATIONS OF A FLOATING BODY.

In all the treatises on Hydrostatics in use in the University, the vertical and angular oscillations of floating bodies are treated of in separate propositions, on the supposition that either of the two kinds of oscillations can take place without the other. It is shewn in Webster's *Theory of Fluids*, Art. 69, that unless the vertical line through the centres of gravity of the body and of the fluid displaced by it in the position of equilibrium passes through the centre of gravity of the plane of floatation, that is, unless the body be symmetrical with respect to this line, there cannot be angular oscillations unattended with vertical, and it might as readily be shewn that there cannot be vertical oscillations unattended with angular. It is not difficult, as will be seen, to determine the simultaneous oscillations of both kinds, provided that the angular oscillations are confined to one plane, namely that passing through the three centres of gravity above mentioned, which may evidently be the case when the body is symmetrical with respect to this plane.



AB is the projection of the plane of floatation in equilibrium, ab the plane of floatation at any time, C the centre of gravity of the plane AB, $a'b'$ parallel to ab through C, G the centre of gravity of the body. Let x be the vertical distance of the centre of gravity of the body below

its place of rest at the time t, θ the angular deviation from the position of equilibrium. Then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{M' - M}{M}g,$$

$$\frac{d^2\theta}{dt^2} = \frac{M'y}{Mk^2}g,$$

M being the mass of the body, M' that of the fluid displaced, y the horizontal distance of the centre of gravity of the fluid displaced from that of the body, Mk^2 the moment of inertia of the body about its centre of gravity.

We must express approximately M' and $M'y$ in terms of x and θ. Let Gh be perpendicular to AB, Ck to ab, and let K be the area of the section AB, a the distance Ch, α the angle CGh, ρ the density of the fluid.

Then $M' = \rho \times \text{vol. } aDb = \rho \times (\text{vol. } a'Db' + \text{vol. } aa'\,b'b)$,

and $\text{vol. } aDb' = \text{vol. } ADB$,

(Webster's *Theory of Fluids*, Art. 68);

therefore $\rho \times \text{vol. } a'Db' = M$.

The vol. $aa'\,b'b = K \times Ck$ nearly:

now x is the sum of the distances through which the centre of gravity is depressed both by the angular and vertical motions. In the figure the centre of gravity is raised by the angular motion, therefore

$$\begin{aligned} x &= Ck - CG\,\{\cos\alpha - \cos(\alpha + \theta)\} \\ &= Ck - \theta\,.\,CG \sin\alpha, \text{ nearly}; \end{aligned}$$

therefore
$$Ck = x + a\theta, \qquad [168]$$

$$M' = M + \rho K(x + a\theta),$$

$$\frac{d^2x}{dt^2} = -\frac{\rho K g}{M}(x + a\theta) \;\ldots\ldots\ldots\ldots\; (1).$$

Again, $M'y$ = moment of aDb = moment $a'Db'$ + moment $a'a'\,bb$. As is proved in all treatises on Hydrostatics in determining the equilibrium of a floating body,

$$\text{moment } a'Db' = (\rho I - Mb)\,\theta,$$

I being the moment of inertia of the plane AB about an axis through C perpendicular to the plane ADB, and b the distance of the centre of gravity of the fluid displaced *below* that of the body.

The moment of $aa'\,b'b = \rho K \times Ck \times CG \sin(\alpha + \theta)$ nearly

$$= \rho K(x + a\theta)\,a \text{ nearly},$$

therefore
$$M'y = \rho K a x + \{\rho(I + Ka^2) - Mb\}\,\theta;$$

hence $$\frac{d^2\theta}{dt^2} = -\frac{\rho Kag}{Mk^2}x - \frac{\{\rho(I + Ka^2) - Mb\}g}{Mk^2}.\theta \;\ldots\ldots (2),$$

(1) and (2) are simultaneous equations which may be solved in the usual way, and thus x and θ expressed in terms of t. It will be convenient to assume

$$\frac{\rho Kg}{M} = m^2, \quad \text{and} \quad \frac{\rho(I + Ka^2) - Mb}{\rho Ka} = c,$$

then they become

$$\frac{d^2x}{dt^2} + m^2(x + a\theta) = 0,$$

$$\frac{d^2\theta}{dt^2} + \frac{m^2a}{k^2}(x + c\theta) = 0.$$

S. S. G.

NEW ANALYTICAL METHODS OF SOLVING CERTAIN GEOMETRICAL PROBLEMS.

THE problems which will form the subject of this Article, are some in which it is required to shew, that the points of intersection of certain lines, straight or curved, described according to given conditions, lie in some other assigned line. The methods of solution will be best explained in applying them to particular propositions.

The first that we shall prove, is one from the Senate-House Problems of the present year: namely,

If on any three chords drawn through the same point in the circumference of a circle, as diameters three circles will be described, the points of intersection of these circles, two and two, lie in the same straight line.

[169] Let a be the radius of the principal circle, α, β, γ the angles which the chords make with the diameter which meets them; let the point through which the chords are drawn be taken for the origin, and the diameter of the principal circle for the axis of x.

The length of the chord whose inclination is α, is $2a \cos \alpha$, and the coordinates of its middle point are $a(\cos \alpha)^2$, $a \cos \alpha \sin \alpha$: hence the equation to the circle described upon it is

$$\{x - a(\cos \alpha)^2\}^2 + \{y - a \cos \alpha \sin \alpha\}^2 = a^2 (\cos \alpha)^2,$$

or $$x^2 + y^2 - 2ax(\cos \alpha)^2 - 2ay \cos \alpha \sin \alpha = 0 \;\ldots\ldots (1).$$

Let $\tan\alpha$, $\tan\beta$, $\tan\gamma = \lambda, \mu, \nu$ respectively; then (1) becomes

$$\rho^2(1+\lambda^2) - 2ax - 2ay\lambda = 0 \;\ldots\ldots\ldots\; (2),$$

putting $x^2 + y^2 = \rho^2$ for abbreviation. Similarly the equation to the second circle is

$$\rho^2(1+\mu^2) - 2ax - 2ay\mu = 0 \;\ldots\ldots\ldots\; (3).$$

Subtracting (2) from (1), and dividing by $\lambda - \mu$, we have

$$\rho^2(\lambda+\mu) - 2ay = 0 \;\ldots\ldots\ldots\ldots\; (4).$$

Subtracting (1) multiplied by μ, from (2) multiplied by λ, and dividing by $\lambda - \mu$,

$$\rho^2(1-\lambda\mu) - 2ax = 0 \;\ldots\ldots\ldots\ldots\; (5).$$

Since these last two equations are obtained by combining (2) and (3), they will be satisfied by the coordinates of the points of intersection of the circles represented by those equations. The next object is to find a relation between $\lambda + \mu$, $\lambda\mu$, and symmetrical functions of all the three quantities λ, μ, ν. For this purpose, let

$$\xi^3 - p\xi^2 + q\xi - r = 0$$

be the equation whose roots are λ, μ, ν; then

$$\lambda^3 - p\lambda^2 + q\lambda - r = 0,$$

$$\mu^3 - p\mu^2 + q\mu - r = 0.$$

Performing on these equations the same processes as before on (2) and (3), we obtain

$$\lambda^2 + \lambda\mu + \mu^2 - p(\lambda+\mu) + q = 0,$$

or

$$(\lambda+\mu)^2 - \lambda\mu - p(\lambda+\mu) + q = 0,$$

and

$$\lambda\mu(\lambda+\mu) - p\lambda\mu + r = 0.$$

Substituting in these two last equations the values of $\lambda + \mu$ and $\lambda\mu$ derived from (4) and (5), we have

$$\frac{4a^2y^2}{\rho^4} - 1 + \frac{2ax}{\rho^2} - p\frac{2ay}{\rho^2} + q = 0,$$

and

$$\left(\frac{2ay}{\rho^2} - p\right)\left(1 - \frac{2ax}{\rho^2}\right) + r = 0:$$

or

$$\frac{4a^2y^2}{\rho^4} + \frac{2a}{\rho^2}(x - py) + q - 1 = 0 \;\ldots\ldots\; (6),$$

and

$$-\frac{4a^2xy}{\rho^4} + \frac{2a}{\rho^2}(px + y) - p + r = 0 \;\ldots\ldots\; (7).$$

These two equations, on account of the manner in [170]
which they were obtained, must both be satisfied by the

coordinates of the points of intersection of the circles represented by (2) and (3). But, since they involve λ, μ, ν symmetrically, they must equally be satisfied by the coordinates of the points of intersection of the two other pairs of circles. And since their number is the same as that of the unknown quantities, x and y, we may, by combining them in various ways, obtain every other equation that is satisfied by those coordinates, and among them, the required relation. This is done by multiplying (6) by x, and (7) by y, and adding; whence, observing that $x^2 + y^2 = \rho^2$, we have

$$(q - 1)\,x - (p - r)\,y + 2a = 0,$$

the equation to a straight line. Substituting for p, q, r their values in terms of the roots, replacing λ, μ, ν by $\tan \alpha$, $\tan \beta$, $\tan \gamma$, and reducing, we find

$$x \cos (\alpha + \beta + \gamma) + y \sin (\alpha + \beta + \gamma) = 2a \cos \alpha, \cos \beta, \cos \gamma.$$

The same method applies to the following problem, of which several solutions were given in the *Philosophical Magazine* in the last two years.

If three tangents be drawn to a parabola, the circle described through their points of intersection will also pass through the focus.

Let λ, μ, ν be the tangents of the angles which the tangents make with the axis, then their equations are

$$y = \lambda x + \frac{m}{\lambda}$$

$$y = \mu x + \frac{m}{\mu}$$

$$y = \nu x + \frac{m}{\nu}.$$

Subtracting the 2nd from the 1st, and dividing by $\lambda - \mu$,

$$0 = x - \frac{m}{\lambda\mu}.$$

Multiplying the 1st and 2nd by λ and μ respectively, subtracting, and dividing by $\lambda - \mu$,

$$y = (\lambda + \mu)\,x.$$

Now if $$\xi^3 - p\xi^2 + q\xi - r = 0$$

be the equation whose roots are λ, μ, ν, we have, as in the former problem,

$$(\lambda + \mu)^2 - \lambda\mu - p\,(\lambda + \mu) + q = 0,$$

and $(\lambda + \mu - p)\,\lambda\mu + r = 0.$

Substituting the values of $\lambda + \mu$ and $\lambda\mu$ in terms of x and y,

$$y^2 - mx - pxy + qx^2 = 0 \;\ldots\ldots\ldots\ldots\; (1),$$

$$my - mpx + rx^2 = 0 \;\ldots\ldots\ldots\ldots\; (2).$$

We must combine these equations so as to obtain [171] the equation to a circle. For this purpose, first multiply (2) by y and (1) by m, and subtract; then, after dividing by x,

$$rxy - mqx + m^2 = 0 \;\ldots\ldots\ldots\ldots\; (3).$$

Next, multiply (1) by r and (3) by p, and add, in order to eliminate the term xy, then

$$ry^2 - rqx^2 - m(r + pq)\,x + m^2p = 0 \;\ldots\ldots\; (4).$$

Finally, multiply (2) by $1 - q$, and add it to (4), therefore

$$r(y^2 + x^2) + m(1 - q)\,y - m(p + r)\,x + m^2p = 0,$$

which is the equation to a circle. It is easy to see that it is satisfied by the values $y = 0$, $x = m$, therefore it passes through the focus.

If a, β, γ be the angles which the tangents make with the axis, and for p, q, r be substituted their values in terms of $\tan a$, $\tan \beta$, $\tan \gamma$, after putting the equation under the form

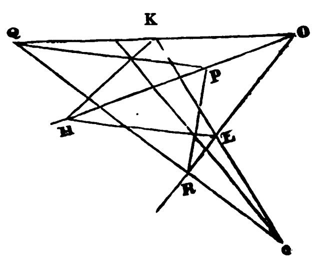
$$r(y^2 + x^2 - m^2) + m(1 - q)\,y - m(p + r)(x - m) = 0,$$

it becomes

$$\sin a \sin \beta \sin \gamma\,(y^2 + x^2 - m^2) + m \cos(a + \beta + \gamma)\,y$$
$$- m \sin(a + \beta + \gamma)(x - m) = 0.$$

The succeeding problems are of a somewhat different nature, and therefore the same method is not applicable to them; but that which we shall employ, is, we believe, in some degree new. The first is from the Senate-House papers of this year.

HKL, *PQR* are two triangles, prove that if the straight lines *HP*, *KQ*, *LR* meet in one point, the intersections of *KL*, *QR*; *LH*, *RP*; *HK*, *PQ*, lie in a straight line.



Let O be the point in which *HP*, *KQ*, *LR* meet, and take it for the origin of coordinates; let ab, $a'b'$, $a''b''$, be rectangular coordinates of H, K, L, and AB, $A'B'$, $A''B''$ of P, Q, R.

Then, since O, H, P are in a straight line,

$$\frac{A}{a} = \frac{B}{b}.$$

Let this ratio be α, and similarly, let

$$\frac{A'}{a'} = \frac{B'}{b'} = \alpha', \quad \text{and} \quad \frac{A''}{a''} = \frac{B''}{b''} = \alpha''.$$

The equation to HK is

$$(b - b')\,x - (a - a')\,y = a'b - ab' \,\ldots\ldots\ldots (1),$$

and that to PQ,

$$(B - B')\,x - (A - A')\,y = A'B - AB',$$

or

$$(\alpha b - \alpha' b')\,x - (\alpha a - \alpha' a')\,y = \alpha\alpha'\,(a'b - ab') \ldots (2).$$

Let

$$nx - my = 1 \,\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

be the equation to a line passing through the intersection [172] of (1) and (2); then, eliminating x and y by cross multiplication, we obtain the following equation between m and n,

$$\{\alpha\alpha'\,(b - b') - (\alpha b - \alpha' b')\}\,m - \{\alpha\alpha'\,(a - a') - (\alpha a - \alpha' a')\}\,n$$
$$+ \alpha - \alpha' = 0 \,\ldots\ldots (4).$$

The same line will also pass through the intersections of the other pairs of lines, if the same values of m and n satisfy these two other equations

$$\{\alpha'\alpha''\,(b' - b'') - (\alpha' b' - \alpha'' b'')\}\,m - \{\alpha'\alpha''\,(a' - a'') - (\alpha' a' - \alpha'' a'')\}\,n$$
$$+ \alpha' - \alpha'' = 0 \,\ldots\ldots (5),$$

$$\{\alpha''\alpha\,(b'' - b) - (\alpha'' b'' - \alpha b)\}\,m - \{\alpha''\alpha\,(a'' - a) - (\alpha'' a'' - \alpha a)\}\,n$$
$$+ \alpha'' - \alpha = 0 \,\ldots\ldots (6);$$

but this is the case, for whether we add (4), (5), (6) as they stand, or after multiplying them by α, α', α'' respectively, in both ways we get the same relation, whence it follows that any one of the equations is a consequence of the other two. To find the value of m, multiply by $\alpha''a''$, αa, $\alpha' a'$, and add, then n disappears, and we find

$$m = \{(\alpha'' - \alpha')\,\alpha a + (\alpha - \alpha'')\,\alpha' a' + (\alpha' - \alpha)\,\alpha'' a''\}$$

divided by

$$[\alpha\alpha'\alpha''\,\{a\,(b' - b'') + a'\,(b'' - b) + a''\,(b - b')\} - (\alpha' b' - \alpha'' b'')\,\alpha a$$
$$- (\alpha'' b'' - \alpha b)\,\alpha' a' - (\alpha b - \alpha' b')\,\alpha'' a''].$$

In like manner,

$$n = \{(\alpha'' - \alpha')\,\alpha b + (\alpha - \alpha'')\,\alpha' b' + (\alpha' - \alpha)\,\alpha'' b''\}$$

divided by

$$[\alpha\alpha'\alpha''\,\{a\,(b' - b'') + a'\,(b'' - b) + a''\,(b - b')\} - (\alpha' b' - \alpha'' b'')\,\alpha a$$
$$- (\alpha'' b'' - \alpha b)\,\alpha' a' - (\alpha b - \alpha' b)\,\alpha'' a''] —$$

If the tangents drawn to every two or three unequal circles be produced till they meet, the points of intersection will be in a straight line.

Let $$(x-a_1)^2+(y-\beta_1)^2=\gamma_1^2$$
be the equation to one of the circles. The equation to a tangent line making an angle θ with the axis of x, is
$$(x-a_1)\sin\theta-(y-\beta_1)\cos\theta=\gamma_1\,\ldots\ldots\ldots(1).$$
If the same line touch another circle whose equation is
$$(x-a_2)^2+(y-\beta_2)^2=\gamma_2^2,$$
we must have
$$a_1\sin\theta-\beta_1\cos\theta+\gamma_1=a_2\sin\theta-\beta_2\cos\theta+\gamma_2,$$
or $$(a_1-a_2)\sin\theta-(\beta_1-\beta_2)\cos\theta+\gamma_1-\gamma_2=0\;\ldots\;(2),$$
which will give two values of θ.

To find the point of intersection of the two tangents, we observe that if x and y in (1) belong to that point, that equation must be satisfied by both the values of θ given by (2). Therefore (1) and (2) must be identical, whence
$$\frac{x-a_1}{\gamma_1}=\frac{a_1-a_2}{\gamma_1-\gamma_2},\quad\frac{y-\beta_1}{\gamma_1}=\frac{\beta_1-\beta_2}{\gamma_1-\gamma_2},\qquad[173]$$
and $$x=\frac{a_2\gamma_1-a_1\gamma_2}{\gamma_1-\gamma_2},\quad y=\frac{\beta_2\gamma_1-\beta_1\gamma_2}{\gamma_1-\gamma_2}.$$
Let $$mx+ny=1$$
be the equation to a line passing through this point, then
$$(a_2\gamma_1-a_1\gamma_2)\,m+(\beta_2\gamma_1-\beta_1\gamma_2)\,n=\gamma_1-\gamma_2\;\ldots\;(3),$$
and the same line will pass through the intersections of the other pairs of tangents, if values of m and n exist which satisfy also the two equations
$$(a_3\gamma_2-a_2\gamma_3)\,m+(\beta_3\gamma_2-\beta_2\gamma_1)\,n=\gamma_2-\gamma_1\;\ldots\ldots\;(4),$$
$$(a_1\gamma_3-a_3\gamma_1)\,m+(\beta_1\gamma_3-\beta_3\gamma_1)\,n=\gamma_3-\gamma_1\;\ldots\ldots\;(5);$$
but this is true, as is seen by multiplying (3), (4), (5) by γ_3, γ_1, γ_2, and adding, when the whole will vanish. The value of m is found by multiplying by β_3, β_1, β_2, and adding; and that of n by multiplying by a_3, a_1, a_2. Thus we find the equation to be
$$\{\beta_1(\gamma_2-\gamma_3)+\beta_2(\gamma_3-\gamma_1)+\beta_3(\gamma_1-\gamma_2)\}\,x$$
$$-\{a_1(\gamma_2-\gamma_3)+a_2(\gamma_3-\gamma_1)+a_3(\gamma_1-\gamma_2)\}\,y$$
$$=(a_2\beta_3-a_3\beta_2)\,\gamma_1+(a_3\beta_1-a_1\beta_3)\,\gamma_2+(a_1\beta_2-a_2\beta_1).$$

S. S. G.

ON THE INTEGRATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

In the present article we shall apply to Simultaneous Linear Differential Equations the principles, the applications of which we have developed in the preceding Numbers of this Journal. The usual method for solving these equations was first given by D'Alembert, and has received but little improvement since his time, although it is so long and tedious, that some change was highly desirable. The process which we shall give here is at once simple and direct, and shews the advantage of recurring frequently to the principles on which our calculations are founded. The theory of the method is sufficiently simple. Since we have shewn that the symbols of differentiation are subject to the same laws of combination as those of number, they may be always treated in the same manner if the coefficients be all constants, which is the only case we shall consider. We have therefore only to separate the symbol of differentiation from its subject, and then proceed to eliminate one of the variables between the given equations, exactly as if the symbol of differentiation were an ordinary coefficient. Thus the difficulty of elimina-
[174] tion becomes reduced to that between ordinary and algebraical equations; and all the facilities for effecting it which have been invented for these in particular cases, may be used for differential equations. When the equations involve various orders of differentials of a variable as well as multiples of it, the symbols of differentiation and of number must be grouped together as one coefficient for the variable. Thus, if we had

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\frac{d^2y}{dt^2} + c\frac{dy}{dt} + mx + my = T,$$

$$\frac{d^2y}{dt^2} + a'\frac{dy}{dt} + b'\frac{d^2x}{dt^2} + c'\frac{dx}{dt} + m'y + n'z = T';$$

they are to be written

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + m\right)x + \left(b\frac{d^2}{dt} + c\frac{d}{dt} + n\right)y = T,$$

$$\left(\frac{d^2}{dt^2} + a'\frac{d}{dt} + m'\right)y + \left(b\frac{d^2}{dt^2} + c'\frac{d}{dt} + n'\right)x = T',$$

and then treated as if each symbol of operation were a factor.

We shall proceed to give some examples; and first let us take

$$\frac{dx}{dt} + ay = 0,$$

$$\frac{dy}{dt} + bx = 0.$$

We have to separate $\frac{d}{dt}$ from the variable, and eliminate one of the variables y or x, as would be done if $\frac{d}{dt}$ were an ordinary constant. This will be done if we multiply the first equation by $\frac{d}{dt}$ and the second by a, and subtract, when we obtain

$$\left(\frac{d^2}{dt^2} - ab\right) x = 0.$$

(It is to be observed, that the word "multiply" is used, not because the operation is really multiplication, since that is a numerical operation, but because it bears a close analogy to multiplication, and is represented symbolically in the same manner.)

Having now eliminated y, we may integrate the equation in x at once. The result is

$$x = c_1 \varepsilon^{(ab)^{\frac{1}{2}} t} + c_2 \varepsilon^{-(ab)^{\frac{1}{2}} t}.$$

And from the first equation we get

$$y = c_2 \left(\frac{b}{a}\right)^{\frac{1}{2}} \varepsilon^{-(ab)^{\frac{1}{2}} t} - c_1 \left(\frac{b}{a}\right)^{\frac{1}{2}} \varepsilon^{(ab)^{\frac{1}{2}} t}.$$

As another example, take the equations [175]

$$\frac{dx}{dt} + ax + by = 0,$$

$$\frac{dy}{dt} + a_1 x + b_1 y = 0;$$

which may be put under the form

$$a\left(\frac{d}{dt} + a\right) x + by = 0,$$

$$a_1 x + \left(\frac{d}{dt} + b_1\right) y = 0.$$

Multiply the first by $\frac{d}{dt} + b_1$, and the second by b, and subtract;

then $$\left\{\left(\frac{d}{dt}+a\right)\left(\frac{d}{dt}+b_1\right)-a_1b\right\}x=0.$$

The coefficient of x is obviously of the second order. Let it be made up of two factors, so that

$$\left(\frac{d}{dt}+h\right)\left(\frac{d}{dt}+k\right)x=0,$$

where $$h+k=a+b_1,\quad hk=ab_1-a_1b;$$

so that they are the roots of the equation

$$z^2-(a+b_1)z+ab_1-a_1b=0.$$

Integrating in the usual manner,

$$x=c_1\varepsilon^{-ht}+c_2\varepsilon^{-kt};$$

and from the first equation we deduce

$$y=\frac{h-a}{b}c\varepsilon^{-ht}+\frac{k-a}{b}c_1\varepsilon^{-kt}.$$

It will make no difference in the method of the solution if there be a function of t on the other side of the equation. As, for instance, we have

$$\frac{dx}{dt}+4x+3y=t,$$

$$\frac{dy}{dt}+2x+5y=\varepsilon^t,$$

or $$\left(\frac{d}{dt}+4\right)x+3y=t,$$

$$\left(\frac{d}{dt}+5\right)y+2x=\varepsilon^t.$$

Eliminating y,

$$\left\{\left(\frac{d}{dt}+4\right)\left(\frac{d}{dt}+5\right)-6\right\}x=1+5t-3\varepsilon^t,$$

[176] or $$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+7\right)x=1+5t-3\varepsilon^t.$$

Integrating with regard to $\dfrac{d}{dt}+7$,

$$\left(\frac{d}{dt}+2\right)x=\tfrac{2}{49}+\tfrac{5}{7}t-\tfrac{3}{8}\varepsilon^t+c_1\varepsilon^{-7t},$$

and $$x=-\tfrac{31}{196}+\tfrac{5}{14}t-\tfrac{1}{8}\varepsilon^t+c_1\varepsilon^{-7t}+c_2\varepsilon^{-2t},$$

the value of y will be found to be

$$y = \tfrac{9}{98} - \tfrac{1}{7}t + \tfrac{5}{24}\varepsilon^{t} - \tfrac{8}{3}\varepsilon^{-7t} - c_2\varepsilon^{-2t}.$$

Take also the equations

$$\frac{dx}{dt} + 5x + y = \varepsilon^{t},$$

$$\frac{dy}{dt} + 3y - x = \varepsilon^{2t}.$$

Eliminating y, we obtain

$$\left(\frac{d}{dt} + 4\right)^{2} x = 4\varepsilon^{t} - \varepsilon^{2t};$$

whence $$x = \left(\frac{d}{dt} + 4\right)^{-2} (4\varepsilon^{t} - \varepsilon^{2t}) + (c_1 t + c_2)\,\varepsilon^{-4t},$$

or $$x = \tfrac{4}{25}\varepsilon^{t} - \tfrac{1}{36}\varepsilon^{2t} + (c_1 t + c_2)\,\varepsilon^{-4t},$$

and $$y = \tfrac{1}{25}\varepsilon^{t} + \tfrac{7}{36}\varepsilon^{2t} + (c_1 t + c_2)\,\varepsilon^{-4t}.$$

If there be three simultaneous equations, the same method is to be pursued, but the calculations must be necessarily long, as in ordinary elimination. We may, however, avail ourselves of the method of cross multiplication. If, for instance, we have

$$\frac{dx}{dt} + by + cz = 0,$$

$$\frac{dy}{dt} + ax + c'z = 0,$$

$$\frac{dz}{dt} + a'x + b'y = 0,$$

or $$\frac{d}{dt}x + by + cz = 0 \;\ldots\ldots\ldots\ldots\; (1),$$

$$ax + \frac{d}{dt}y + c'z = 0 \;\ldots\ldots\ldots\ldots\; (2),$$

$$a'x + b'y + \frac{d}{dt}z = 0 \;\ldots\ldots\ldots\ldots\; (3),$$

we may eliminate y and z by multiplying [177]

$$(1) \text{ by } \frac{d^2}{dt^2} - b'c', \quad (2) \text{ by } b'c - b\frac{d}{dt}, \quad (3) \text{ by } bc' - c\frac{d}{dt},$$

and the result will be

$$\left\{\frac{d}{dt}\left(\frac{d^2}{dt^2} - b'c'\right) + a\left(b'c - b\frac{d}{dt}\right) + a'\left(bc' - c\frac{d}{dt}\right)\right\} x = 0,$$

or $$\left\{\frac{d^3}{dt^3} - (ab + a'c + b'c')\frac{d}{dt} + ab'c + a'bc'\right\} x = 0,$$

which can be integrated by the usual method. The equations for determining y and z will, of course, be symmetrical with that for x.

There will be three arbitrary constants arising from the integration of this equation; and it would appear, that as there are other two similar equations, from which six more arbitrary constants would arise, there would be nine on the whole. But there are really only three independent arbitrary constants, as we are able to deduce the other two variables y and z from the value of the first, without integration, and consequently the arbitrary constants in their expressions must be derivable from those in the first integral.

The same method of integration may of course be applied to any number of simultaneous differential equations: but the difficulty of elimination rises rapidly with the number of variables, as in the case of ordinary numerical equations. We shall take as an example of four simultaneous equations, those given by Mr. Airy for determining the secular variations of the eccentricity and longitude of the perihelion, *Plan. Theor.* p. 123.

They are of the form

$$0 = \frac{d}{dt}u + a_1 v - a_2 v' \ldots\ldots\ldots\ldots (1),$$

$$0 = a_1 u - \frac{d}{dt}v - a_2 u' \ldots\ldots\ldots\ldots (2),$$

$$0 = \frac{d}{dt}u' + b_1 v' - b_2 v \ldots\ldots\ldots\ldots (3),$$

$$0 = b_1 u' - \frac{d}{dt}v' - b_2 u \ldots\ldots\ldots\ldots (4).$$

To reduce them to three simultaneous equations, eliminate v by multiplying (1) by $\frac{d}{dt}$ and (2) by a_1, and adding. Then

$$\left(\frac{d^2}{dt^2} + a_1^2\right) u - a_1 a_2 u' - a_2 \frac{d}{dt} v' = 0 \ldots\ldots\ldots (5).$$

Again, eliminate the same variable between (1) and (3), by multiplying (1) by b_2, and (3) by a_1, and adding, so that

$$b_2 \frac{d}{dt}u + a_1 \frac{d}{dt}u' + (a_1 b_1 - a_2 b_2) v' = 0 \ldots\ldots\ldots (6),$$

and the equation (4) is [178]

$$b_2 u - b_1 u' + \frac{d}{dt} v' = 0 \ldots\ldots\ldots\ldots (7).$$

Eliminating u' and v' by cross multiplication between (5), (6), and (7), we obtain an equation which reduces itself to

$$\left\{\frac{d^4}{dt^4} + (a_1^2 + b_1^2 + 2a_2 b_2)\frac{d^2}{dt^2} + a_1 b_1 - a_2 b_2\right\} u - 0 \ldots . (8),$$

which we shall have no difficulty in integrating.

From its form we see that the operating factor is decomposable into two binomial quadratic factors, so that we may put it under the form

$$\left(\frac{d^2}{dt^2} + k_1^2\right)\left(\frac{d^2}{dt^2} + k_2^2\right) u = 0,$$

$-k_1^2$, $-k_2^2$ being the roots of the equation

$$z^2 + (a_1^2 + b_1^2 + 2a_2 b_2) z + a_1 b_1 - a_2 b_2 = 0, \ldots\ldots (9).$$

Integrating with respect to the first factor, as in page 28 of our first Number,

$$\left(\frac{d^2}{dt^2} + k_2^2\right) u = c_1 \cos k_1 x + c_2 \sin k_1 x,$$
$$= c_1 \cos (k_1 x + a),$$

by changing the form of the constants; whence

$$u = \left(\frac{d^2}{dt^2} + k_2^2\right)^{-1} \{c_1 \cos (k_1 x + a_1)\} + c_2 \cos (k_2 x + a_2).$$

The operating factor will only affect the constant of its subject, and as that is arbitrary, we may write

$$u = c_1 \cos (k_1 x + a_1) + c_2 \cos (k_2 x + a_2).$$

To find the value of v, we may observe, that if we had eliminated v' instead of v at first, we should have, instead of (6) and (7), the similar equations

$$a_2 \frac{d}{dt} u' + b_1 \frac{d}{dt} u + (a_1 b_1 - a_2 b_2) v = 0,$$
$$a_2 u' - a_1 u + \frac{d}{dt} v = 0.$$

Eliminating u' between these by multiplying the second by $\frac{d}{dt}$, and subtracting,

$$\left\{\frac{d^2}{dt^2} - (a_1 b_1 - a_2 b_2)\right\} v = (a_1 + b_1) \frac{du}{dt};$$

or putting for u its value, and effecting the differentiation indicated,

$$\left\{\frac{d^2}{dt^2} - (a_1b_1 - a_2b_2)\right\} v$$

$$= -(a_1 + b_1)\{c_1k_1 \sin(k_1x + \alpha_1) + c_2k_2 \sin(k_2x + \alpha_2)\};$$

[179] whence, putting $a_1b_1 - a_2b_2 = A$,

$$v = (a_1 + b_1)\left\{\frac{c_1k_1}{k_1^2 + A} \sin(k_1x + \alpha_1) + \frac{c_2k_2}{k_2^2 + A} \sin(k_2x + \alpha_2)\right\}.$$

By substituting the actual expressions for k_1 and k_2, this may be simplified. For solving equation (9), we get

$$k_1^2 = \tfrac{1}{2}\{a_1^2 + b_1^2 + 2a_2b_2 - (a_1 + b_1)[(a_1 - b_1)^2 + 4a_2b_2]^{\frac{1}{2}}\}:$$

then $k_1^2 + A = \tfrac{1}{2}(a_1 + b_1)\{a_1 + b_1 - [(a_1 - b_1)^2 + 4a_2b_2]^{\frac{1}{2}}\}$.

But we have also

$$k_1 = \tfrac{1}{2}\{a_1 + b_1 - [(a_1 - b_1)^2 + 4a_2b_2]^{\frac{1}{2}}\},$$

and similarly for k_2; so that

$$v = c_1 \sin(k_1x + \alpha_1) + c_2 \sin(k_2x + \alpha_2).$$

Knowing u and v, we easily determine u' and v'. For, by equation (2),

$$a_2u' = a_1u - \frac{d}{dt}v$$

$$= (a_1 - k_1)c_1 \cos(k_1x + \alpha_1) + (a_1 - k_2)c_2 \cos(k_2x + \alpha_2);$$

similarly v' may be found.

In this case we have had an example of the integration of simultaneous differential equations, of an order higher than the first. We shall take another in

$$\frac{d^2y}{dt^2} - az - by = c,$$

$$\frac{d^2z}{dt^2} - a'z - b'y = c',$$

being two of the equations for determining the circumstances of the movement of a floating body in a position nearly of equilibrium, the other two being

$$\frac{d^2\phi}{dt^2} + n\phi - m = 0,$$

$$\frac{d^2s}{dt^2} - g\frac{d^2\phi}{dt^2} = 0.$$

Putting the equations under the form

$$\left(\frac{d^2}{dt^2} - b\right) y - az = c,$$

$$\left(\frac{d^2}{dt^2} - a'\right) z - b'y = c',$$

we can eliminate z by multiplying the first by $\frac{d^2}{dt^2} - a'$, and the second by a, and adding, when we get

$$\left(\frac{d^2}{dt^2} - a'\right)\left(\frac{d^2}{dt^2} - b\right) y - ab'y = ac' - ca'.$$

If k_1^2, k_2^2 be the roots of [180]

$$z^2 - (a' + b) z + a'b - ab' = 0,$$

the equation becomes

$$\left(\frac{d^2}{dt^2} - k_1^2\right)\left(\frac{d^2}{dt^2} - k_2^2\right) y = ac' - ca';$$

the integral of which is

$$y = \frac{ac' - ca'}{k_1^2 k_2^2} + c_1 \varepsilon^{k_1 t} + c_2 \varepsilon^{-k_1 t} + c_3 \varepsilon^{k_2 t} + c_4 \varepsilon^{-k_2 t},$$

$$y = \frac{ac' - ca'}{a'b - ab'} + c_1 \varepsilon^{k_1 t} + c_2 \varepsilon^{-k_1 t} + c_3 \varepsilon^{k_2 t} + c_4 \varepsilon^{-k_2 t}.$$

The value of z may be found from that of y; and if we know the initial circumstances, we can determine the arbitrary constants.

The same method may be extended to simultaneous partial differential equations, according to the principles developed in Article IX. of our third Number. Take, for instance, the equations

$$\frac{dz}{dx} + c\frac{du}{dx} + a\frac{dz}{dy} + bz = 0,$$

$$c'\frac{dz}{dx} + \frac{du}{dx} + a\frac{du}{dy} + bu = 0,$$

which may be written

$$\left(\frac{d}{dx} + a\frac{d}{dy} + b\right) z + c\frac{d}{dx} u = 0,$$

$$c'\frac{d}{dx} z + \left(\frac{d}{dx} + a\frac{d}{dy} + b\right) u = 0.$$

To eliminate u, multiply the first by $\left(\dfrac{d}{dx} + a\dfrac{d}{dy} + b\right)$, and the second by $c'\dfrac{d}{dx}$, and subtract. Then

$$\left\{\frac{d^2}{dx^2} + \frac{2\left(a\dfrac{d}{dy} + b\right)}{1 - cc'}\frac{d}{dx} + \frac{\left(a\dfrac{d}{dy} + b\right)^2}{1 - cc'}\right\} z = 0.$$

If we put $1 - \sqrt{(cc')} = m_1$, $1 + \sqrt{(cc')} = m_2$, this may be resolved into factors,

$$\left\{\frac{d}{dx} + m_1\left(a\frac{d}{dy} + b\right)\right\}\left\{\frac{d}{dx} + m_2\left(a\frac{d}{dy} + b\right)\right\} z = 0;$$

the integral of which is

$$z = \epsilon^{-m_1 x\left(a\frac{d}{dy} + b\right)}\phi(y) + \epsilon^{-m_2 x\left(a\frac{d}{dy} + b\right)}\psi(y),$$

or
$$z = \epsilon^{-m_1 bx}\phi(y - am_1 x) + \epsilon^{-m_2 bx}\psi(y - am_2 x).$$

Mr. Airy, at page 279 note of the Undulatory Theory,
[181] gives as the equations for determining the small disturbances of an elastic medium in three dimensions,

$$\frac{d^2u}{dt^2} = a^2\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

$$\frac{d^2v}{dt^2} = a^2\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

$$\frac{d^2w}{dt^2} = a^2\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

We might eliminate v and w, by cross multiplication, between these equations, and so obtain an equation in u which might be integrated, but it will be more convenient to proceed as follows. Let

$$r = a^2\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right);$$

then
$$\frac{d^{-1}}{dx^{-1}}\frac{d^2u}{dt^2} = \frac{d^{-1}}{dy^{-1}}\frac{d^2v}{dt^2} = \frac{d^{-1}}{dz^{-1}}\frac{d^2w}{dt^2} = r.$$

Multiply by $\dfrac{d^2}{dx^2}$, $\dfrac{d^2}{dy^2}$, $\dfrac{d^2}{dz^2}$, and add; then

$$\frac{d^2}{dt^2}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) r,$$

or
$$\left\{\frac{d^2}{dt^2} - a^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\right\} r = 0\,;$$
the integral of which is
$$r = \varepsilon^{at\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)^{\frac{1}{2}}} \phi(x, y, z) + \varepsilon^{-at\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)^{\frac{1}{2}}} \psi(x, y, z):$$
knowing r, we can determine u, v, and w, since
$$u = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dx}, \quad v = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dy}, \quad w = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dz}.$$
This solution is due to Mr. Greatheed.

We have now given a sufficient number of examples to enable the student to understand thoroughly the method, and we think that they shew clearly the advantages of a process, which, to some persons, might appear to carry out to a startling extent the principles on which it is founded.

D. F. G.

[182]

ANALYTICAL SOLUTIONS OF SOME PROBLEMS IN PLANE ASTRONOMY.

When it has been required to determine the ratio of the small variations of arcs or angles on the surface of a sphere, mathematicians have generally employed a geometrical method, probably because it appeared to them that the application of the Differential Calculus would lead to laborious processes. But it will be shewn in this Article, that Analytical solutions of such problems are not necessarily tedious, and the method by which they are abridged will be seen to be very similar in all.

1. When the latitude and hour-angle are determined from two altitudes of the Sun, and the time between, to find the errors caused by given small errors in the observed altitudes.

Let l be the latitude, z one of the observed zenith distances, h the hour-angle from noon, δ the Sun's declination, a his south azimuth. Then
$$\cos z = \sin l \sin \delta + \cos l \cos \delta \cos h.$$
Differentiate, using Δ to denote small finite variations, which will be nearly in the ratio of the differentials; and considering δ constant, as the Sun's declination is supposed to be known accurately: therefore
$$-\sin z \Delta z = (\cos l \sin \delta - \sin l \cos \delta \cos h)\,\Delta l - \cos l \cos \delta \sin h \Delta h.$$

Now $\quad \cos l \sin \delta - \sin l \cos \delta \cos h = -\cos a \sin z,$

and $\quad \cos \delta \sin h = \sin z \sin a;$

substituting and dividing by $\sin z$,

$$\Delta z = \cos a \Delta l + \cos l \sin a \Delta h.$$

If z', a' be the zenith distance and azimuth at the second observation,

$$\Delta z' = \cos a' \Delta l + \cos l \sin a' \Delta h;$$

therefore
$$\Delta l = \frac{\Delta z \sin a' - \Delta z' \sin a}{\sin (a' - a)},$$

and
$$\Delta h = \frac{\Delta z' \cos a - \Delta z \cos a'}{\cos l \sin (a' - a)}.$$

2. To find the errors caused by parallax in the hour-angle and declination of a heavenly body.

The notation remaining the same, we have the equation

$$\cot h \sin a = \cos l \cot z + \sin l \cos a,$$

in which h and z are the only variables. Differentiating,

$$\frac{\sin a}{(\sin h)^2} \Delta h = \frac{\cos l}{(\sin z)^2} \Delta z.$$

[183] But if P be the horizontal parallax,

$$\Delta z = P \sin z,$$

and
$$\sin a = \frac{\cos \delta \sin h}{\sin z};$$

therefore
$$\Delta h = P \frac{\cos l \sin h}{\cos \delta}.$$

Again,
$$-\cos a = \frac{\sin \delta - \sin l \cos z}{\cos l \sin z},$$

or
$$\cos a \cos l = \sin l \cot z - \sin \delta \operatorname{cosec} z.$$

Differentiating,

$$0 = \{\sin \delta \operatorname{cosec} z \cot z - \sin l (\operatorname{cosec} z)^2\} \Delta z - \cos \delta \operatorname{cosec} z \Delta \delta,$$

$$0 = P(\sin \delta \cos z - \sin l) - \cos \delta \Delta \delta,$$

but $\quad \cos z = \sin l \sin \delta + \cos l \cos \delta \cos h;$

therefore

$$\cos \delta \Delta \delta = P \{\sin l (\sin \delta)^2 + \cos l \sin \delta \cos \delta \cos h - \sin l\},$$

and $\quad \Delta \delta = P (\cos l \sin \delta \cos h - \sin l \cos \delta).$

3. To find the precession of a star in right ascension and declination.

Let a, δ, l, λ, be the right ascension, declination, longitude, and latitude of the star, and ω the obliquity of the ecliptic.

Then $\quad \sin \delta = \cos \omega \sin \lambda + \sin \omega \cos \lambda \sin l$;

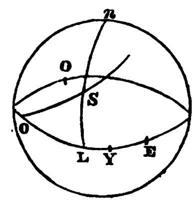
therefore $\cos \delta \, \Delta\delta = \sin \omega \cos \lambda \cos l \, \Delta l$.

But $\quad \cos \lambda \cos l = \cos \delta \cos a$;

therefore
$$\Delta\delta = \sin \omega \cos a \, \Delta l,$$
$$= 50'', 18.t. \sin \omega \cos a,$$
t being the interval expressed in years.

Again, $\quad \sin \lambda = \cos \omega \sin \delta - \sin \omega \cos \delta \sin a$;

therefore
$$0 = (\cos \omega \cos \delta + \sin \omega \sin \delta \sin a)\,\Delta\delta - \sin \omega \cos \delta \cos a \, \Delta a.$$
Substituting the value of $\Delta\delta$,
$$\Delta a = 50'', 18.t. (\cos \omega + \sin \omega \tan \delta \sin a).$$



4. To find the aberration of a star in longitude and latitude.

Let S be a star, O a point in the ecliptic 90° behind the Earth's plane, SL perpendicular to the ecliptic.

Let l, λ be the latitude and longitude of the star. Then
$$\Upsilon O = 180^\circ + \odot - 90^\circ = 90^\circ + \odot,$$
$$LO = 90^\circ + \odot - l.$$
By Napier's rules,
$$\cos O = \cot OS \tan LO,$$
$$= -\cot OS \cot (\odot - l).$$
Taking the logarithmic differentials, [184]
$$0 = \frac{\Delta . OS}{\sin OS \cos OS} - \frac{\Delta l}{\sin (\odot - l) \cos (\odot - l)}.$$
But $\Delta . OS = n \sin OS$; n denoting the ratio of the Earth's velocity to that of light, and
$$\cos OS = \cos SL . \cos LO = -\cos \lambda \sin (\odot - l).$$
Hence
$$\Delta l = -n \frac{\cos (\odot - l)}{\cos \lambda}.$$
Again,
$$\sin \lambda = \sin O . \sin OS;$$
therefore
$$\cot \lambda \, \Delta\lambda = \cot OS . \Delta . OS,$$
$$= n \cos OS,$$
$$= -n \cos \lambda \sin (\odot - l);$$
therefore
$$\Delta\lambda = -n \sin \lambda \sin (\odot - l).$$

5. The aberration in right ascension and declination may be found in the following manner.

Since $\tan a = \cos\omega \tan l - \sin\omega \sec l \tan\lambda$,

$$(\sec a)^2 \Delta a = \cos\omega (\sec l)^2 \Delta l - \sin\omega \{\sec l \tan l \tan\lambda \Delta l + \sec l (\sec\lambda)^2 \Delta\lambda\}$$

$$= -n \cos\omega (\sec l)^2 \sec\lambda \cos(\odot - l) + n \sin\omega \sec l \sec\lambda \tan\lambda \{\tan l \cos(\odot - l) + \sin(\odot - l)\},$$

$$= -n \cos\omega (\sec l)^2 \sec\lambda \cos(\odot - l) + n \sin\omega (\sec l)^2 \sec\lambda \tan\lambda \sin\odot,$$

$$= -n \frac{\cos\omega \cos\odot + (\cos\omega \tan l - \sin\omega \sec l \tan\lambda) \sin\odot}{\cos l \cos\lambda},$$

$$= -n \frac{\cos\omega \cos\odot + \tan a \sin\odot}{\cos a \cos\delta};$$

therefore $$\Delta a = -n \frac{\cos\omega \cos a \cos\odot + \sin a \sin\odot}{\cos\delta}.$$

Also, since $\sin\delta = \sin\omega \sin l \cos\lambda + \cos\omega \sin\lambda$,

$$\cos\delta \Delta\delta = \sin\omega (\cos l \cos\lambda \Delta l - \sin l \sin\lambda \Delta\lambda) + \cos\omega \cos\lambda \Delta\lambda,$$

$$= -n \sin\omega \{\cos l \cos(\odot - l) - \sin l (\sin\lambda)^2 \sin(\odot - l)\} - n \cos\omega \sin\lambda \cos\lambda \sin(\odot - l)\},$$

$$= -n \sin\omega \cos\odot - n (\sin\omega \sin l \cos\lambda + \cos\omega \sin\lambda) \cos\lambda \sin(\odot - l),$$

$$= -n \sin\omega \cos\odot - n \sin\delta \cos\lambda \sin(\odot - l),$$

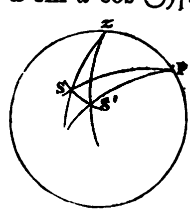
$$= -n \sin\omega \cos\odot - n \sin\delta \cos l \cos\lambda (\sin\odot - \cos\odot \tan l),$$

$$= -n \sin\omega \cos\odot - n \sin\delta \cos a \cos\delta \{\sin\odot - \cos\odot (\cos\omega \tan a + \sin\omega \tan\delta \sec a)\}$$

$$= -n \sin\omega (\cos\delta)^2 \cos\odot - n \sin\delta \cos\delta (\cos a \sin\odot - \cos\omega \sin a \cos\odot);$$

[185] therefore $$\Delta a = -n \{\sin\omega \cos\delta \cos\odot + \sin\delta (\cos a \sin\odot - \cos\omega \sin a \cos\odot)\}.$$

6. To find the time of year at which the Sun passes from one given zenith distance to another in the shortest or longest time.



Let Z be the zenith, P the pole of the equator, S, S' the two positions of the Sun. Let z, z' be the given zenith dis-

tances, h, h' the corresponding hour-angles, δ the Sun's declination, and l the latitude of the place. Then

$$\cos z = \sin l \sin \delta + \cos l \cos \delta \cos h \;\ldots\ldots (1),$$

$$\cos z' = \sin l \sin \delta + \cos l \cos \delta \cos h' \;\ldots\ldots (2),$$

and $h' - h$ is a maximum or minimum; therefore

$$0 = (\sin l \cos \delta - \cos l \sin \delta \cos h)\, d\delta - \cos l \cos \delta \sin h\, dh,$$

$$0 = (\sin l \cos \delta - \cos l \sin \delta \cos h')\, d\delta - \cos l \cos \delta \sin h'\, dh',$$

$$dh' - dh = 0;$$

hence
$$\left(\frac{\sin l \cos \delta - \cos l \sin \delta \cos h'}{\sin h'} - \frac{\sin l \cos \delta - \cos l \sin \delta \cos h}{\sin h}\right) d\delta = 0 \ldots (3);$$

therefore, either $d\delta = 0$, which gives the times of the solstices, or the other factor is equal to zero. Now

$$\frac{\sin l \cos \delta - \cos l \sin \delta \cos h}{\cos l \sin h} = \cot PSZ;$$

therefore $\cot PSZ = \cot PS'Z$,

therefore $\cos PSZ = \pm \cos PS'Z$,

or
$$\frac{\sin l - \cos z \sin \delta}{\sin z \cos \delta} = \pm \frac{\sin l - \cos z' \sin \delta}{\sin z \cos \delta},$$

$$\sin l\,(\sin z' \mp \sin z) = \sin \delta \sin (z' \mp z) \;\ldots\ldots (4).$$

Taking the upper sign, we have

$$2 \sin l \cos \frac{z'+z}{2} \sin \frac{z'-z}{2} = 2 \sin \delta \sin \frac{z'-z}{2} \cos \frac{z'-z}{2};$$

therefore
$$\sin \delta = \sin l \frac{\cos \dfrac{z'+z}{2}}{\cos \dfrac{z'-z}{2}} \;\ldots\ldots\ldots\ldots (5);$$

whence the time of year is known.

If it be required to find the time between the zenith distances, we have from (3)

$$\sin l \cos \delta\,(\sin h' - \sin h) = \cos l \sin \delta \sin (h' - h);$$

therefore
$$\sin l \cos \delta \cos \frac{h'+h}{2} = \cos l \sin \delta \cos \frac{h'-h}{2}.$$

By adding (1) and (2), [186]

$$\cos \frac{z'+z}{2} \cos \frac{z'-z}{2} = \sin l \sin \delta + \cos l \cos \delta \cos \frac{h'+h}{2} \cos \frac{h'-h}{2}.$$

Eliminating $\cos\dfrac{h'+h}{2}$,

$$\sin l\cos\frac{z'+z}{2}\cos\frac{z'-z}{2}=\sin\delta\left\{(\sin l)^2+(\cos l)^2\cos\left(\frac{h'-h}{2}\right)^2\right\}.$$

Substituting the value of $\sin\delta$ from (5),

$$\left(\cos\frac{z'-z}{2}\right)^2=(\sin l)^2+(\cos l)^2\cos\left(\frac{h'-h}{2}\right)^2;$$

therefore $\left(\sin\dfrac{z'-z}{2}\right)^2=(\cos l)^2\left(\sin\dfrac{h'-h}{2}\right)^2$,

$$\sin\frac{h'-h}{2}=\pm\sec l\sin\frac{z'-z}{2}\ \ldots\ldots\ldots\ldots\ (6).$$

To find the results of taking the lower sign in (4), we have only to change the sign of z, which gives

$$\sin\delta=\sin l\,\frac{\cos\dfrac{z'-z}{2}}{\cos\dfrac{z'+z}{2}}\ \ldots\ldots\ldots\ (7),$$

$$\sin\frac{h'-h}{2}=\pm\sec l\sin\frac{z'+z}{2}\ \ldots\ldots\ldots\ (8).$$

Since there are two periods in the year at which the Sun has the same declination, within the same half year, and at equal distances from the equinoxes, equations (5) and (7) give each two periods of maximum or minimum duration, and the solstices are two others, so that there are six in all. But one or both of the equations (5) and (7) may give a value of δ greater than the obliquity of the ecliptic, which, therefore, the Sun never attains, so that there may be only four or two maxima and minima. In the problem of finding the time of longest or shortest twilight, the values of z and z' are 90° and 108°; so that equations (5) and (7) become respectively

$$\sin\delta=-\sin l\tan 9°\ \ldots\ldots\ldots\ldots\ (9),$$

$$\sin\delta=-\sin l\cot 9°\ \ldots\ldots\ldots\ (10).$$

Of these, the former gives a value of δ less than the obliquity, whatever be the latitude. The value for north latitude, 52°, is $-7°.10'.10''$, which occurs on the 2nd or 3rd of March, and the 11th or 12th of October. But (10) will give a value of δ which is never attained, unless l be not greater than [187] $3°.36'.55''$. At the equator the values of δ given by (9) and (10) coincide, being both zero.

After some considerations, which enable us to distinguish between maximum and minimum values, we obtain the following results.

At the equator, the duration of twilight is a minimum at the equinoxes, and a maximum at the solstices.

Between the equator and latitude $3°.\,36'.\,55''$, the duration is a minimum at two periods near the equinoxes, in the winter half of the year, a maximum at two others nearer the winter solstice, a minimum at the winter, and a maximum at the summer solstice.

At the above latitude $3°.36'.55''$, the duration is a minimum at two periods near the equinoxes, in the winter half of the year, and a maximum at the solstices.

The seasons are here spoken of, taking into account their being opposite on different sides of the equator.

S. S. G.

ON CERTAIN CASES OF CONSECUTIVE SURFACES.*

THERE is a very simple theorem, which does not seem to have been much noticed by English writers, but which is of frequent use in Analytical Geometry, particularly when all the equations are put in a symmetrical form. It is, that the formula

$$\frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''} = \dots$$

always implies the truth of the following, viz.

$$\frac{a}{b} = \frac{a'}{b'} = \dots = \frac{a + a' + a'' + \dots}{b + b' + b'' + \dots} = \frac{ma + m'a' + m''a'' + \dots}{mb + m'b' + m''b'' + \dots}$$

$$= \frac{\sqrt{(a^2 + a'^2 + a''^2 + \dots)}}{\sqrt{(b^2 + b'^2 + b''^2 + \dots)}}$$

$m, m', m'', \dots$ being any quantities whatever. The proof is so obvious, that it is unnecessary to put it down here. This being premised, suppose the equation to a surface is

$$u = h \dots\dots\dots\dots\dots\dots (1),$$

where h is an absolute constant, and $u = F(x, y, z, a, b, c)$, a, b, c being parameters, by the variation of which the surface

* From a Correspondent.

[188] takes its different positions. And first, suppose a, b, and c to vary, subject to the single condition

$$f(a, b, c) = k \ldots\ldots\ldots\ldots\ldots (2),$$

which is equivalent to supposing the equation (1) to contain two arbitrary parameters. Now, suppose the partial differential coefficients of u and $f(a, b, c)$, taken with regard to a, b, c, are P, Q, R, A, B, C, respectively. Then, in order to find the intersection of the surface with its consecutives, we shall obviously have to combine the equation (1) with the two following, viz.

$$Pda + Qdb + Rdc = 0,$$

$$Ada + Bdb + Cdc = 0.$$

If between these two equations we eliminate da, we obtain

$$(AQ - BP)\, db + (AR - PC)\, dc = 0;$$

and since there is no relation between db and da, their coefficients in this equation must separately be equal to nothing; that is,

$$AQ - BP = 0, \quad AR - PC = 0,$$

whence we immediately derive

$$\frac{P}{A} = \frac{Q}{B} = \frac{R}{C} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This formula, combined with (1), determines the coordinates of the point in which the surface is intersected by all its consecutives; and if we also combine it with (2), and eliminate a, b, and c, we shall obtain the equation to the envelope. This elimination will often be easy, if the functions u and $f(a, b, c)$ are homogeneous with respect to a, b, and c. For suppose m and n are their respective degrees of homogeneity, and we shall have

$$aP + bQ + cR = mh,$$

$$aA + bB + cC = nk.$$

Hence, applying the theorem above laid down to the formula (3), we get

$$\frac{P}{A} = \frac{Q}{B} = \frac{R}{C} \left(= \frac{aP + bQ + cR}{aA + bB + cC} \right) = \frac{mh}{nk} \ldots\ldots (4),$$

by the help of which a, b, and c may be eliminated from either of the equations (1), (2).

If the generating surface was a plane, suppose its equation was

$$ax + by + cz = h^2 \ldots\ldots\ldots\ldots\ldots (5),$$

and we should have $P = x$, $Q = y$, $R = z$; and hence, substituting in the formula (3) and applying the same theorem, we should get

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = \frac{h^2}{aA + bB + cC} = \frac{\sqrt{(x^2 + y^2 + z^2)}}{\sqrt{(A^2 + B^2 + C^2)}} \;\dots (6).$$

These equations determine immediately the coordinates x, y, z of the point in which the plane is intersected by all its consecutives, and the length of the radius vector r [189] from the origin to this point. Also, if λ, μ, ν are the angles which r makes with the three axes, we have evidently

$$\frac{\cos\lambda}{A} = \frac{\cos\mu}{B} = \frac{\cos\nu}{C} = \frac{1}{\sqrt{(A^2 + B^2 + C^2)}} \;\dots\dots (7).$$

Hence it is easily seen, that if we described a surface whose equation was

$$f(x, y, z) = k \;\dots\dots\dots\dots\dots (8),$$

a, b, and c would always be equal to the coordinates of some point in it; and if a tangent plane was drawn at this point, the perpendicular upon it from the origin would make precisely the same angles with the axes as those indicated by the formula (7), and would therefore coincide with the radius vector r. Moreover, if p was the length of the perpendicular, we should have

$$p = \frac{aA + bB + cC}{\sqrt{(A^2 + B^2 + C^2)}};$$

and therefore it is evident, from the equations (6), that $rp = h^2$. Hence, if in the perpendicular from the origin upon the tangent plane to the surface (8), we take a point, such that its distance from the origin multiplied by the length of the perpendicular shall $= h^2$, the locus of this point will be the envelope of the system of planes defined by the two equations

$$ax + by + cz = h^2, \quad f(a, b, c) = k.$$

As another example, suppose the equation to the surface is

$$\frac{\phi}{a^m} + \frac{\chi}{b^m} + \frac{\psi}{c^m} = 1 \;\dots\dots\dots\dots\dots (9),$$

where ϕ, χ, and ψ are any functions of x, y, z, not containing a, b, c; and suppose the relation between a, b, and c is

$$\left(\frac{\alpha}{a}\right)^n + \left(\frac{\beta}{b}\right)^n + \left(\frac{\gamma}{c}\right)^n = 1 \;\dots\dots\dots (10);$$

here the equations (4) become

$$\frac{\phi}{\alpha^n a^{m-n}} = \frac{\chi}{\beta^n b^{m-n}} = \frac{\psi}{\gamma^n c^{m-n}} = 1 \;\dots\dots\dots (11);$$

and if we take the values of a, b, c from these, and substitute them in (9) or (10), we get the equation to the envelope, viz.

$$\left(\frac{\phi}{\alpha^m}\right)^{\frac{n}{n-m}} + \left(\frac{\chi}{\beta^m}\right)^{\frac{n}{n-m}} + \left(\frac{\psi}{\gamma^m}\right)^{\frac{n}{n-m}} = 1.$$

In the particular case when $n = m$, the equations (11) are reduced to the three $\phi = \alpha^m$, $\chi = \beta^m$, $\psi = \gamma^m$, and the envelope consists of three distinct surfaces defined by these equations. And if ϕ, χ, ψ each contain only one of the variables x, y, z, the envelope is reduced to a certain definite number of points. For instance, all the ellipsoids defined by the two equations

[190] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1,$$

will intersect one another in the eight points, whose coordinates are $\pm\, \alpha$, $\pm\, \beta$, $\pm\, \gamma$.

But if a, b, c were subject to two equations of condition, suppose, for instance,

$$f(a, b, c) = k \ldots\ldots\ldots\ldots\ldots (12),$$

$$f_1(a, b, c) = k_1 \ldots\ldots\ldots\ldots\ldots (13),$$

(which is the same thing as supposing the equation to the surface to contain only one arbitrary parameter,) let A_1, B_1, C_1, be the partial differential coefficients of $f_1(a, b, c)$, the others remaining as before, and we shall have the three equations

$$Pda + Qdb + Rdc = 0,$$

$$Ada + Bdb + Cdc = 0,$$

$$A_1da + B_1db + C_1dc = 0.$$

To eliminate the differentials, apply the method of indeterminate multipliers, and we shall have

$$\left.\begin{aligned} P &= \lambda A + \mu A_1 \\ Q &= \lambda B + \mu B_1 \\ R &= \lambda C + \mu C_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (14).$$

Let $aP + bQ + cR = v$, $aA + bB + cC = S$, $aA_1 + bB_1 + cC_1 = S_1$; then, if we add the equations (14), after having respectively multiplied them by a, b, and c, we get

$$v = \lambda S + \mu S_1;$$

and if we substitute in the equations (14) the value of λ derived from this equation, we obtain

$$SP - Av = \mu\,(A_1S - AS_1),$$

$$SQ - Bv = \mu\,(B_1S - BS_1),$$

$$SR - Cv = \mu\,(C_1S - CS_1)$$

and therefore

$$\frac{SP - Av}{A_1 S - A S_1} = \frac{SQ - Bv}{B_1 S - B S_1} = \frac{SR - Cv}{C_1 S - C S_1} \ldots\ldots(15).$$

This formula, combined with (1), comprehends the equations to the curve in which the surface is intersected by its consecutive; and if we eliminate a, b, c by means of (12) and (13), we shall obtain the equation to the envelope. If the functions u, $f(a, b, c)$, $f_1(a, b, c)$, are homogeneous, and of the degrees n, m, m_1 respectively, then $v = nh$, $S = mk$, and $S_1 = m_1 k_1$.

Suppose, for example, it was required to find the surface generated by the consecutive intersections of tangent planes drawn to the ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots(16),$$

at all the points where it is cut by the plane, whose [191]
equation is

$$lx + my + nz = k \ldots\ldots\ldots\ldots(17).$$

The equation to a tangent plane at the point (x, y, z), is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} = 1 \ldots\ldots\ldots\ldots(18),$$

ξ, η, ζ being the coordinates of any point in the tangent plane. Here the variable parameters are x, y, z, subject to the two equations (16) and (17). And we have

$$P = \frac{\xi}{a^2}, \quad A = \frac{2x}{a^2}, \quad A_1 = l,$$

&c., $v = 1$, $S = 2$, $S_1 = k$; hence the formula (15) becomes

$$\frac{\xi - x}{la^2 - kx} = \frac{\eta - y}{mb^2 - ky} = \frac{\zeta - z}{nc^2 - kz} \ldots\ldots (19),$$

which is equivalent to the equations to the generating line of the envelope. We may put it under another form, by subtracting 1 from each term, after having multiplied by k; we thus obtain

$$\frac{k\xi - la^2}{la^2 - kx} = \frac{k\eta - mb^2}{mb^2 - ky} = \frac{k\zeta - nc^2}{nc^2 - kz}.$$

This shews that the generating line always passes through a fixed point, whose coordinates are $\frac{la^2}{k}$, $\frac{mb^2}{k}$, $\frac{nc^2}{k}$.

In order to eliminate x, y, z from the equations (19), if we multiply the numerator and denominator of each term by $\frac{\xi}{a^2}, \frac{\eta}{b^2}$, and $\frac{\zeta}{c^2}$ respectively, and then add the numerators and denominators, we obtain an expression which, by the help of equations (17) and (18), immediately reduces to the following, viz:

$$\frac{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1}{l\xi + m\eta + n\zeta - k}.$$

Again, if we multiply by l, m, n, and add, we obtain

$$\frac{l\xi + m\eta + n\zeta - k}{l^2a^2 + m^2b^2 + n^2c^2 - k^2};$$

and these two expressions must be equal, by the principle stated at the beginning. Hence,

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1 + \frac{(l\xi + m\eta + n\zeta - k)^2}{l^2a^2 + m^2b^2 + n^2c^2 - k^2},$$

which is the equation to the conical surface required.

If it was required to find the locus of the vertex of this cone, when the plane which cuts the ellipsoid is moved, but made to pass through a fixed point (α, β, γ), suppose x, y, z are the coordinates of the vertex, and we have seen above that we shall have

[192]
$$\frac{x}{la^2} = \frac{y}{mb^2} = \frac{z}{nc^2} = \frac{1}{k} \; \ldots\ldots\ldots\ldots \; (20);$$

also, α, β, γ must satisfy the equation to the cutting plane, that is,

$$l\alpha + m\beta + n\gamma = k;$$

multiplying, then, the terms of this last equation by the terms of (20), we have

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

which is the equation to the locus sought, and represents a plane. If the point (α, β, γ) is a point in the surface of the ellipsoid, this equation coincides with the equation to the tangent plane at that point, as it evidently ought.

M. N. N.

MATHEMATICAL NOTE.

Geometrical Theorem.—Let $A_1A_2A_3\ldots$ be a polygon of n sides inscribed in a circle, a_1, a_2, a_3, &c., the angles which the sides A_1A_2, A_2A_3, &c. subtend at the centre. Then

$$\angle A_1A_2A_3 = \pi - A_1A_5A_3,$$

$$= \pi - \frac{a_1 + a_2}{2},$$

$$\angle A_3A_4A_5 = \pi - \frac{a_3 + a_4}{2},$$

&c. &c.

$$\angle A_{n-1}A_nA_1 = \pi - \frac{a_{n-1} + a_n}{2}.$$

If n be even, adding all these together, we get

$$\angle A_1A_2A_3 + \angle A_3A_4A_5 + \text{\&c.} + \angle A_{n-1}A_nA_1$$

$$= \frac{n}{2}\pi - \frac{2\pi}{2} = (n-2)\frac{\pi}{2},$$

or the sum of the alternate angles is equal to $n - 2$ right angles, a curious extension of Euclid, III. 22.

γ.

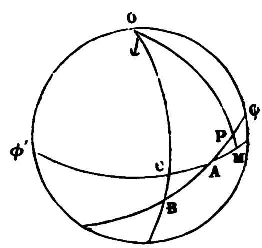
[193]

ON THE METHOD OF SPHERICAL COORDINATES.

In two very ingenious papers "On the equations to loci on the surface of the sphere," published in the 12th Volume of the *Edinburgh Transactions*, Mr. Davies has investigated the nature and properties of curves traced on the surface of the sphere, by referring them to spherical coordinates. Although the analogy with the ordinary methods of Analytical Geometry might appear to be pretty obvious, yet, till the publication of these papers, no systematic attempt had been made to apply the method of coordinates to spherical curves; one or two isolated problems only having been solved by this means. Nor do these papers appear to have received much attention, since we do not find that the methods there developed have yet been transferred to any elementary books. We think, therefore, that it will be acceptable to our readers, to lay before them a sketch of the principles and applications of the method; not going into all the minutiæ of the subject,

but still giving a sufficient number of examples to illustrate the processes. Those who are desirous of following out the subject to a farther extent, will find the original papers in the *Edinburgh Transactions* both interesting and instructive, and well worth their perusal. We shall not in all cases adhere closely to the methods employed by Mr. Davies, but shall change them where we think an alteration will render the treatment of the subject simpler.

As might be expected, this method bears a close analogy to Analytical Geometry of two dimensions, only substituting great circles of the sphere for straight lines. For as a [194] straight line is the simplest line which can be drawn on a plane, so a great circle is the simplest line which can be drawn on a spherical surface. In order, therefore, to determine the position of a point on a sphere, we assume two great circles intersecting each other at right angles for coordinate axes, the point of intersection being the origin.



Thus, let $C\phi$, $C\psi$ be two great circles intersecting each other at right angles, and if P be any point on the surface of the sphere, its position with reference to C will be completely determined by drawing PM an arc of a great circle, passing through P, and perpendicular to CM; for knowing the axes CM and CP, we can find P. CM, PM are the spherical coordinates of P, and we shall generally represent them by ϕ and ψ. As in Analytical Geometry, we shall assume axes measured in the directions $C\phi$ and $C\psi$ to be positive, and those measured in the directions $C\phi'$, $C\psi'$ to be negative. This is not absolutely necessary, since a negative arc ϕ' is equal to a positive arc $2\pi - \phi'$, which may therefore be used in its place, and the results will be the same.

We may also refer spherical curves to polar coordinates, though there is not nearly so much distinction between the two methods in spherical as in rectilinear Geometry. If we produce the arc MP, it will meet the axis of ψ in a point O, which is the pole of the great circle, which we have taken as the axis of ϕ. If, then, we take O as the origin, OP as the radius vector, and COP as the angle vector, the position of the point P may be determined by means of a relation between these two quantities. It will be seen at once from the figure, that the angle vector is equal to ϕ in

the other method, and that the radius vector is the complement of ψ. There is, therefore, very little difference in expressions for curves by either method, and the one can be easily transferred to the other. Mr. Davies has generally made use of polar coordinates: in the following pages we shall chiefly employ the other.

We shall now proceed to the investigation of equations to lines traced on the sphere, beginning with a great circle.

1. To find the equation to a great circle. Let PAB be a great circle cutting the axes in A and B. Let $CM = \phi$, $PM = \psi$, $CA = a$, $CB = \beta$, $PA\phi = \iota$.

Then, by Napier's rules,

$$\sin AM = \tan PM . \cot PAM;$$

therefore $$\tan \psi = -\tan \iota \sin (a - \phi);$$

when $$\phi = 0, \quad \psi = \beta;$$

therefore $$\tan \beta = -\tan \iota \sin a:$$

whence $$\tan \psi = \frac{\tan \beta \sin (a - \phi)}{\sin a} = -\frac{\tan \beta}{\sin a} \sin (\phi - a).$$

The general equation to a great circle may therefore be put under the form

$$\tan \psi = m \sin (\phi - a) \ldots\ldots\ldots\ldots (1),$$

where $$m = -\frac{\tan \beta}{\sin a}.$$ [195]

If it pass through a given point ϕ_1, ψ_1, we have

$$\tan \psi_1 = m \sin (\phi_1 - a);$$

whence $$\tan \psi = \frac{\tan \psi_1}{\sin (\phi_1 - a)} \sin (\phi - a)$$

is the equation to a great circle passing through a given point ϕ_1, ψ_1.

If it pass through two given points, the same method may be used, but the following is more convenient.

The general equation

$$\tan \psi = m \sin (\phi - a)$$

may be put under the form

$$\tan \psi = A \sin \phi + B \cos \phi,$$

or the still more general one

$$\tan \psi = A \sin (\phi - a) + B \sin (\phi - \beta);$$

whence we have

$$\tan \psi_1 = A \sin (\phi_1 - \alpha) + B \sin (\phi_1 - \beta),$$
$$\tan \psi_2 = A \sin (\phi_2 - \alpha) + B \sin (\phi_2 - \beta),$$

$\psi_1, \phi_1, \psi_2, \phi_2$ being the coordinates of the two given points.

As we have four indeterminate constants and only two equations, we may assume $\alpha = \phi_1$, $\beta = \phi_2$;

therefore $\qquad \tan \psi_1 = - B \sin (\phi_2 - \phi_1),$

and $\qquad \tan \psi_2 = \ \ A \sin (\phi_2 - \phi_1);$

therefore $\quad A = \dfrac{\tan \psi_2}{\sin (\phi_2 - \phi_1)}, \quad$ and $B = \dfrac{\tan \psi_1}{\sin (\phi_2 - \phi_1)}.$

Hence the equation to a great circle passing through two given points $\phi_1, \psi_1, \phi_2, \psi_2$, is

$$\tan \psi \sin (\phi_2 - \phi_1) = \tan \psi_2 \sin (\phi - \phi_1) - \tan \psi_1 \sin (\phi - \phi_2).$$

2. To find the coordinates of the intersection of two arcs.

Let the equation be

$$\tan \psi = m_1 \sin (\phi - \alpha_1) = m_1 (\sin \phi \cos \alpha_1 - \cos \phi \sin \alpha_1),$$
$$\tan \psi = m_2 \sin (\phi - \alpha_2) = m_2 (\sin \phi \cos \alpha_2 - \cos \phi \sin \alpha_2).$$

When the coordinates are common, we have, by dividing one by the other,

$$1 = \frac{m_1}{m_2} \, \frac{\tan \phi \cos \alpha_1 - \sin \alpha_1}{\tan \phi \cos \alpha_2 - \sin \alpha_2},$$

whence $\quad \tan \phi = \dfrac{m_1 \sin \alpha_1 - m_2 \sin \alpha_2}{m_1 \cos \alpha_1 - m_2 \cos \alpha_2};$

therefore $\sin \phi = \dfrac{m_1 \sin \alpha_1 - m_2 \sin \alpha_2}{\{m^2 - 2m_1m_2 \cos (\alpha_1 - \alpha_2) + m_2^2\}^{\frac{1}{2}}}.$

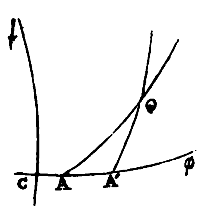
[196] and $\cos \phi = \dfrac{m_1 \cos \alpha_1 - m_2 \cos \alpha_2}{\{m_1^2 - 2m_1m_2 \cos (\alpha_1 - \alpha_2) + m_2^2\}^{\frac{1}{2}}}.$

Substituting then in the expression for $\tan \psi$, we find, after reduction,

$$\tan \psi = \frac{m_1m_2 \sin (\alpha_1 - \alpha_2)}{\{m_1^2 - 2m_1m_2 \cos (\alpha_1 - \alpha_2) + m_2^2\}^{\frac{1}{2}}}.$$

3. To find the angle between two great circles whose equations are given.

Let AQ, $A'Q$ be the circles, and let their equations be



$$\tan \psi = m_1 \sin (\phi - \alpha_1),$$
$$\tan \psi' = m_2 \sin (\phi - \alpha_2).$$

Now in the triangle AQA',

$\cos AQA = \cos AA' \sin QAA' \sin QA'A - \cos QAA \cos QA'A$,

and $AA' = a_1 - a_2$, $\tan QAA' = m_1$, $\tan QA'A = -m_2$;

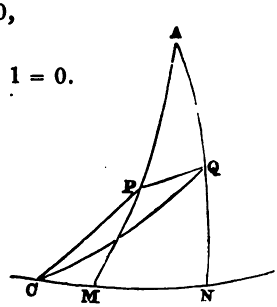
whence $$\cos AQA' = \frac{m_1 m_2 \cos(a_1 - a_2) + 1}{\sqrt{(1+m_1^2)}\sqrt{(1+m_2^2)}}.$$

If the circles cut each other at right angles,

$$\cos AQA' = 0,$$

and the condition is

$$m_1 m_2 \cos(a_1 - a_2) + 1 = 0.$$



4. To find the length of the arc joining two points in terms of the coordinates of the points.

Let PQ be the points whose coordinates are ϕ_1, ψ_1, ϕ_2, ψ_2. Produce MP, NQ to meet in A the pole of CMN. Then

$$\cos PQ = \sin AP \sin AQ \cos PAQ + \cos AP \cos AQ,$$

or $\cos PQ = \cos\psi_1 \cos\psi_2 \cos(\phi_2 - \phi_1) + \sin\psi_1 \sin\psi_2$.

Hence the equation to a small circle, whose radius is the radius of the sphere multiplied by $\sin\gamma$, or whose distance from its pole is γ, and the coordinates of whose centre are ϕ_1, ψ_1, is

$$\cos\psi_1 \cos\psi \cos(\phi - \phi_1) + \sin\psi_1 \sin\psi = \cos\gamma.$$

5. To find the equation to the perpendicular from a given point on a given great circle.

Let the equation to the given great circle be

$$\tan\psi = m \sin(\phi - a),$$

ϕ_1, ψ_1 the coordinates of the given point. Then, if the required equation be of the form

$$\tan\psi = m_1 \sin(\phi - a_1),$$

we find $$m_1 = \frac{\tan\psi_1}{\sin(\phi_1 - a_1)},$$

and $$\frac{m \tan\psi_1}{\sin(\phi_1 - a_1)} \cos(a_1 - a) + 1 = 0,$$

as a condition for determining a_1 by (3). Whence [197]

$$m \tan\psi_1 (\cos a_1 \cos a + \sin a_1 \sin a)$$
$$= -(\sin\phi_1 \cos a_1 - \cos\phi_1 \sin a_1),$$

which gives $\tan \alpha_1 = \dfrac{\sin \phi_1 + m \tan \psi_1 \cos \alpha}{\cos \phi_1 - m \tan \psi_1 \sin \alpha}.$

Therefore

$$\tan \psi = \frac{\tan \psi_1 \sin (\phi - \alpha_1)}{\sin (\phi_1 - \alpha_1)} = \tan \psi_1 . \frac{\sin \phi - \tan \alpha_1 \cos \phi}{\sin \phi_1 - \tan \alpha_1 \cos \phi_1}$$

$$= \tan \psi_1 . [(\cos \phi_1 - m \tan \psi_1 \sin \alpha) \sin \phi - (\sin \phi_1 + m \tan \psi_1 \cos \alpha) \cos \phi]$$

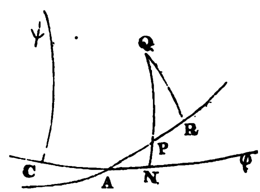
divided by

$$[(\cos \phi_1 - m \tan \psi_1 \sin \alpha) \sin \phi_1 - (\sin \phi_1 + m \tan \psi_1 \cos \alpha) \cos \phi_1];$$

whence, after reduction,

$$\tan \psi = - \frac{1}{m \cos (\phi_1 - \alpha)} \{\sin (\phi - \phi_1) - m \tan \psi_1 \cos (\phi - \alpha)\}.$$

6. To find the length of the perpendicular from a given point on a given great circle.



Let AR be the circle whose equation is

$$\tan \psi = m \sin (\phi - \alpha),$$

Q the given point whose coordinates are ϕ_1, ψ_1.

Then $\tan PN = m \sin (\phi_1 - \alpha)$,

also $\sin QR = \sin PQ . \sin QPR$,

but $\sin QPR = \sin APN$,

and $\cos PAN = \sin APN \cos PN$,

so that $\sin QPR = \dfrac{\cos PAN}{\cos PN}$;

therefore

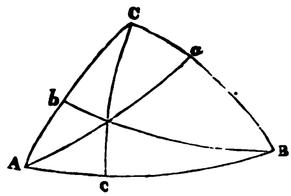
$$\sin QR = \frac{\sin PQ \cos PAN}{\cos PN}$$

$$= \frac{\sin (QN - PN) \cos PAN}{\cos PN}$$

$$= \sin QN - \cos QN \tan PN \cos PAN$$

$$= \frac{\sin \psi_1 - m \cos \psi_1 \sin (\phi - \alpha)}{\sqrt{(1 + m^2)}}.$$

7. The equations we have arrived at serve to demonstrate readily that the perpendiculars on the sides of a spherical triangle from the opposite angles meet in one point.



Let AB, Cc be the axes, $cA = \alpha$, $cB = \beta$. The equation to AC, since it cuts the axis at a distance $-\alpha$, and is inclined to it at an angle A, is

$$\tan\psi = \tan A \sin(\phi + \alpha).$$

[198] The equation to BC which cuts the axis at a distance β, and is inclined to it at an angle $\pi - \beta$, is

$$\tan\psi = -\tan\beta \sin(\phi - \beta).$$

The equation to Bb is by (3) making $\phi_1 = \beta$, $\psi_1 = 0$,

$$\tan\psi = -\frac{1}{\tan A \cos(\alpha + \beta)} \sin(\phi - \beta).$$

Similarly the equation to Aa is

$$\tan\psi = \frac{1}{\tan B \cos(\alpha + \beta)} \sin(\phi + \alpha).$$

Put $\phi = 0$ in these equations, and we have

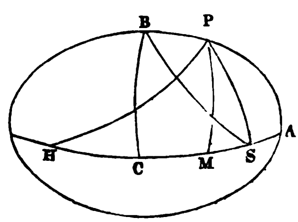
$$\tan\psi_1 = \frac{\sin\beta}{\tan A \cos(\alpha + \beta)}$$

$$\tan\psi_2 = \frac{\sin\alpha}{\tan B \cos(\alpha + \beta)}.$$

But $\quad \sin\alpha = \cot A \tan Cc, \quad \sin\beta = \cot B \tan Cc.$

Therefore the values of $\tan\psi_1$, $\tan\psi_2$ are identical, each being equal to $\dfrac{\cot A \cot B \tan Cc}{\cos AB}$.

8. We shall now proceed to the consideration of some more complicated spherical curves, beginning with one which, being defined in the same manner as a plane ellipse, is called a spherical ellipse.



To find the locus of a point, the sum of whose distances from two given points is constant.

Let S, H be the two fixed points; take the arc joining them for the axis of ϕ, and the middle point of it for the origin.

Let $SH = 2\gamma$, $SP + HP = 2a$, $CM = \phi \,.\, PM = \psi$.

Then
$$\cos SP = \cos(\gamma - \phi)\cos\psi$$
$$\cos HP = \cos(\gamma + \phi)\cos\psi.$$

Adding and reducing,

$$\cos \tfrac{1}{2}(HP - SP) = \frac{\cos \gamma \cos \phi \cos \psi}{\cos \alpha}.$$

Similarly, by subtraction,

$$\sin \tfrac{1}{2}(HP - SP) = \frac{\sin \gamma \sin \phi \cos \psi}{\sin \alpha}.$$

Squaring and adding,

$$\left\{\left(\frac{\cos \gamma \cos \phi^2}{\cos \alpha}\right) + \left(\frac{\sin \gamma \sin \phi}{\sin \alpha}\right)^2\right\}(\cos \psi)^2 = 1,$$

or $(\sec \psi)^2 = \left(\dfrac{\cos \gamma}{\cos \alpha}\right)^2 - \dfrac{(\cos \gamma \sin \alpha)^2 - (\sin \gamma \cos \alpha)^2}{(\cos \alpha \sin \alpha)^2}(\sin \phi)^2.$

[199] Therefore

$$(\tan \psi)^2 = \frac{(\cos \gamma)^2 - (\cos \alpha)^2}{(\cos \alpha)^2} - \frac{(\sin \alpha)^2 - (\sin \gamma)^2}{(\cos \alpha \sin \alpha)^2}(\sin \phi)^2$$

$$= \frac{(\sin \alpha)^2 - (\sin \gamma)^2}{(\cos \alpha \sin \alpha)^2}\{(\sin \alpha)^2 - (\sin \phi)^2\},$$

when $\phi = 0$ $(\tan \psi)^2 = \dfrac{(\sin \alpha)^2 - (\sin \gamma)^2}{(\cos \alpha)^2} = (\tan \beta)^2$ suppose.

Then $\quad (\tan \psi)^2 = \left(\dfrac{\tan \beta}{\sin \alpha}\right)^2 \{(\sin \alpha)^2 - (\sin \phi)^2\},$

or

$$\left(\frac{\tan \psi}{\tan \beta}\right)^2 + \left(\frac{\sin \phi}{\sin \alpha}\right)^2 = 1,$$

which is the final equation.

In the spherical, as in the plane ellipse, the distance between the focus and the extremity of the axis minor is equal to the semi-axis major, for we have

$$(\tan \beta)^2 = \frac{(\sin \alpha)^2 - (\sin \gamma)^2}{(\cos \alpha)^2};$$

whence $\quad \sec \beta = \dfrac{\cos \gamma}{\cos \alpha},$

and $\quad \cos \alpha = \cos \beta \cos \gamma.$

But $\quad \cos \beta \cos \gamma = \cos SB,$

and therefore $\quad SB = \alpha = CA.$

The spherical ellipse and hyperbola are the same. For if the sum of the distances of P from S and H be constant, the difference of the distances of P from S and a point opposite to H is also constant, and therefore the locus of P is also the locus of a curve traced according to the definition of a hyperbola.

9. The locus of the vertex of a right-angled spherical triangle whose base is given is a spherical ellipse.

Take the middle point of the base as origin, the axis of ϕ coinciding with the base, and let the length of the base be $2a$. The equations to the two sides will be

$$\tan\psi = m_1 \sin(\phi - a)$$
$$\tan\psi = m_2 \sin(\phi + a),$$

and since they are at right angles to each other,

$$m_1 m_2 \cos 2a + 1 = 0 \text{ by } (3).$$

Therefore $$(\tan\psi)^2 = -\frac{\sin(\phi + a)\sin(\phi - a)}{\cos 2a},$$

or $$(\tan\psi)^2 = \frac{(\sin a)^2 - (\sin\phi)^2}{\cos 2a},$$

which is the equation to an ellipse.

If β be the value of ψ when $\phi = 0$, [200]

$$(\tan\beta)^2 = \frac{(\sin a)^2}{\cos 2a},$$

whence $$\sin\beta = \tan a.$$

10. The curve of intersection of an ellipsoid with a sphere is a spherical ellipse.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Put $x = r\cos\phi\cos\psi$, $y = r\sin\phi\cos\psi$, $z = r\sin\psi$.

Then $$(\cos\psi)^2\left\{\frac{(\sin\phi)^2}{b^2} + \frac{(\cos\phi)^2}{a^2}\right\} + \frac{(\sin\psi)^2}{c^2} = \frac{1}{r^2}.$$

Dividing by $(\cos\psi)^2$, and putting $1 - (\sin\phi)^2$ for $(\cos\phi)^2$,

$$\frac{(\tan\psi)^2}{c^2} + (\sin\phi)^2\left\{\frac{1}{b^2} - \frac{1}{a^2}\right\} + \frac{1}{a^2} = \frac{1}{r^2} + \frac{(\tan\psi)^2}{r^2}.$$

Therefore

$$(\tan\psi)^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) + (\sin\phi)^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \frac{1}{r^2} - \frac{1}{a^2}.$$

Since, in order that the surfaces may intersect, we must have $r < a$ and $> c$; this is the equation to a spherical ellipse whose semi-axes are

$$\sin^{-1}\left(\frac{\frac{1}{r^2} - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}}\right)^{\frac{1}{2}} \text{ and } \tan^{-1}\left(\frac{\frac{1}{r^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{r^2}}\right)^{\frac{1}{2}}.$$

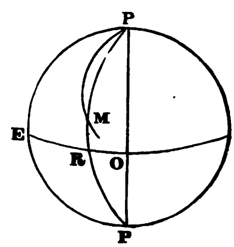
The section will be a great circle when the major axis becomes equal to π, or $\frac{1}{r^2} - \frac{1}{a^2} = \frac{1}{b^2} - \frac{1}{a^2}$, or $r = b$, in which case the equation becomes

$$\tan \psi = \left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}} \cos \phi \,;$$

therefore the tangent of the inclination of this circle to that of ϕ is

$$\left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}}.$$

11. The next curve we shall consider is the equable spherical spiral, of which a particular case is the spiral of Pappus. Its definition is: If a meridian PRP' revolve [201] uniformly about an axis PP' of a sphere, while a point M moves from P uniformly along PRP' from P to P', the locus of the point M is the equable spherical spiral. Take P as the origin of the polar coordinates, EPM as the angle vector $= \theta$, PM the radius vector $= \phi$, and let the ratio of the motions be $m:n$; then the required equation is



$$m\phi = n\theta.$$

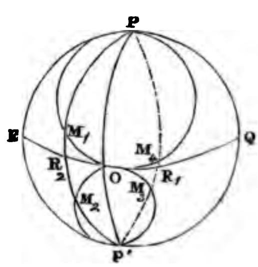
To discuss this equation, we shall assume different relations between m and n.

Let, in the first place, $m = n$; then $\phi = \theta$.

During the first quadrant the point M_1 will be in the spherical octant PEO, and at the end of the first quadrant it will be at O.

During the second quadrant, it will be in the octant $OP'Q$, and at π of longitude it will be at P'.

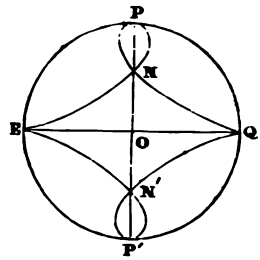
During the third quadrant the radius vector will be measured on a meridian $PR_1P'M_3$, of which PR_1P' is on the posterior surface of the sphere; but as ϕ is also greater than π, the point M_3 will be on the convex



side of the sphere in the octant EOP'; and at $\frac{3\pi}{2}$, M_3 will be at O.

During the fourth quadrant it will be in the octant POQ, and at the end will return to P, and after that it will retrace the same path.

When $m = 2n$, or $\phi = \frac{1}{2}\theta$, the accompanying diagram represents the locus, where ON, ON', are equal to $\frac{\pi}{4}$.



When $m = 4n$, or $\phi = \frac{1}{4}\theta$, we have the particular case considered by Pappus. Here ϕ does not acquire the value $\frac{\pi}{2}$, or the curve does not cut the equator till after a complete revolution of the meridian. After two revolutions it is at the opposite pole, and after four revolutions it returns to the origin.

Generally, if m and n be commensurable, the branches of the spiral will return in the same order, and coalesce when $\theta = 2mn\pi$, but if they be incommensurable this will never occur.

12. If a cylinder, whose radius is half the radius of a sphere, and the centre of whose base is placed at the distance of half the radius from the centre, intersect the sphere, the curve of intersection is that equable spiral whose equation is

$$\phi = \theta.$$

The equation to the cylinder is (the radius of the sphere being 1)

$$y^2 + (x - \tfrac{1}{2})^2 = \tfrac{1}{4};$$

where it meets the sphere

$$y = \cos\theta \sin\phi, \quad x = \cos\theta\cos\phi;$$

therefore $$(\cos\theta)^2 - \cos\theta\cos\phi + \tfrac{1}{4} = \tfrac{1}{4}; \qquad [202]$$

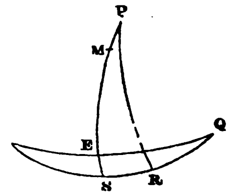
whence $$\cos\theta = \cos\phi,$$

and $$\theta = \pm\phi.$$

13. To find the path of the vertical projection of the Sun, supposing the Sun to move round the earth in a circle with an equable motion.

Let EQ be the equator, SQ the ecliptic, P, M their respective poles. Then, in the right-angled triangle PSR, considering the angle at S as the right angle,

$$\cos PR = \cos SP \,.\, \cos SR.$$



Let $PR = \phi$, PM = obliquity of ecliptic $= \omega$;

then $\qquad \cos \phi = - \sin \omega \cos SR.$

Now, if n be the ratio of the Earth's angular velocity about its axis to that about the Sun, $\theta = nSR$,

therefore $$\cos \phi = - \sin \omega \cos \frac{\theta}{n}$$

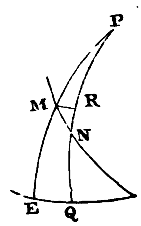
is the equation to the path.

If $\omega = \frac{\pi}{2}$, this becomes the equable spherical spiral.

14. The last curve which we shall consider, is the Rhumb line or Loxodrome, which has always attracted much attention from its use in navigation. The definition of this curve is, that it cuts all the meridians at equal angles.

Let P be the pole, PE, PQ two successive meridians, making an angle $d\theta$ with each other, θ being the longitude of PM measured from a given point. Let $PM = \phi$. Draw MR parallel to the equator: then $RN = d\phi$: hence $MR = \sin \phi d\theta$. Then, considering the ultimate elements of the arcs which form the elemental triangle MNR, as straight lines, we have

$$\tan MNR = \frac{MR}{RN}.$$



As the angle MNR is to be the same for every meridian, let it $= \alpha$. Then

$$\frac{\sin \phi d\theta}{d\phi} = \tan \alpha,$$

or $$\frac{d\phi}{\sin \phi} = \cot \alpha d\theta;$$

whence integrating,

$$\log \tan \frac{\phi}{2} = \theta \cot \alpha + C.$$

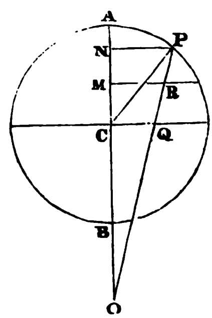
To determine C. Let $\phi = \frac{\pi}{2}$ when $\theta = 0$. Then $C = 0$.

The equation may therefore be put under the form [203]

$$\tan\frac{\phi}{2} = \varepsilon^{\theta\cot\alpha}.$$

15. It is frequently an interesting problem to find what are the plane curves produced by the projections of spherical curves. For this purpose we shall investigate a general formula for such transformation.

Let A be the pole to which the curves have been referred, C the centre of the circle, P a point in the curve, O any point situate in the axis AB, from which the point P is to be projected on a plane MR parallel to the equator.



Let $AC = r$, $CO = a$, $CM = b$, $AP = \phi$, $MR = \rho$.

It is clear that the projection of the angle between two meridians will be equal to the original angle, since the projecting planes will pass through AO. Hence if we consider the curve on MR as referred to polar coordinates, the angle vector will remain the same, and we have only to determine the radius vector $MR = \rho$ in terms of ψ.

Now, from the similar triangles RMO, PNO,

$$RM : OM = PN : ON,$$

$$\rho : a + b = r\sin\phi : a + r\cos\phi.$$

Whence $$\rho = \frac{(a+b)\,r\sin\phi}{a + r\cos\phi}.$$

We shall confine our attention to the three principal projections: the orthographic, the stereographic, and the gnomonic.

In the orthographic the projecting lines are parallel, therefore a is infinite, and we find

$$\rho = r\sin\phi.$$

In the stereographic $a = r$ and $b = c$, therefore

$$\rho = \frac{r\sin\phi}{1+\cos\phi} = r\tan\frac{\phi}{2}.$$

In the gnomonic $a = 0$, $b = r$, therefore

$$\rho = r\tan\phi.$$

Applying these expressions to the equation to the rhumb line, we have for its orthographic projection

$$\rho = \frac{2r}{\varepsilon^{\theta \cot \alpha} + \varepsilon^{-\theta \cot \alpha}}.$$

For the stereographic

$$\rho = r\varepsilon^{\theta \cot \alpha}.$$

For the gnomonic

$$\rho = \frac{2r}{\varepsilon^{-\theta \cot \alpha} - \varepsilon^{\theta \cot \alpha}},$$

which are three of Cotes's spirals.

[204] There are many other interesting points regarding spherical curves, such as their curvature, rectification, and quadrature: but we shall defer the consideration of these till a future Number.

S. S. G.

ON SOME PROPERTIES OF THE PARABOLA.*

There are many very interesting properties of the Conic Sections which are not to be found in the usual works on the subject, but are scattered through various memoirs in scientific Journals. Those relating to the properties of polygons inscribed in and circumscribed round conic sections, have been investigated by a great many writers both in France and England. Pascal was the first who engaged in these researches, and was led by the curious properties which he discovered to call one of these polygons the "hexagramme mystique." After him Maclaurin gave a proof of a theorem which is not only beautiful in itself, but also very fertile in its consequences. In more recent times Brianchon has demonstrated the remarkable theorems, that in all hexagons either inscribed in or circumscribed round a conic section, the three diagonals joining opposite angles will intersect in one point. Subsequently, Davies in this country, and Dandelin in Belgium, proved in different ways the same propositions along with others. The latter adopted a very peculiar method, deducing these and many other properties of sections of the cone by considering the cone as a particular case of the

* From a Correspondent.

"hyperboloide gauche." Generally speaking the Geometrical method is more easily applied than the Analytical to these cases, and accordingly all the proofs given have depended on geometry, with the exception of one published by Mr. Lubbock in the Number of the *Philosophical Magazine* for August 1838. He has there demonstrated, by analysis, Brianchon's Theorem for a circumscribing hexagon in the particular case where the conic section is a parabola; but his method is tedious, and not remarkable for symmetry and elegance, so that another proof is still desirable. The following one is founded on the form of the equation to the tangent of the parabola which is given in Art. 2 of our first Number.

Let the parabola be referred to its vertex, then the equation to its tangent by that article is

$$y = \frac{x}{a} + ma,$$

where a is the tangent of the angle which the tangent [205]
makes with the axis of y. If a' be the corresponding quantity for another tangent, its equation will be

$$y = \frac{x}{a'} + ma'.$$

Combining these equations, we shall find for the coordinates of the point of intersection of the two tangents

$$x = maa', \quad y = m(a + a').$$

We shall distinguish the tangents which form the different sides of the hexagon by suffixing numbers to the a which determines their position, and we shall likewise distinguish the coordinates of the summits of the hexagon by suffix letters.

The equations to the three diagonals are these:

$$(1) \quad y(a_4a_5 - a_1a_2) - x(a_4 + a_5 - a_1 - a_2) = m\{(a_1 + a_2)a_4a_5 - (a_4 + a_5)a_1a_2\}.$$

$$(2) \quad y(a_5a_6 - a_2a_3) - x(a_5 + a_6 - a_2 - a_3) = m\{(a_2 + a_3)a_5a_6 - (a_5 + a_6)a_2a_3\}$$

$$(3) \quad y(a_6a_1 - a_3a_4) - x(a_6 + a_1 - a_3 - a_4) = m\{(a_3 + a_4)a_1a_6 - (a_1 + a_6)a_3a_4\}.$$

Expressions which, as they ought to be, are symmetrical with respect to the a's.

Multiply (1) by a_6, (2) by $-a_4$, (3) by a_2, and add. Then y will disappear, and we shall find

$$x = m\,\frac{a_3a_6(a_4a_5 - a_1a_2) - a_4a_1(a_5a_6 - a_2a_3) + a_5a_2(a_6a_1 - a_3a_4)}{a_1a_2 - a_2a_3 + a_3a_4 - a_4a_5 + a_5a_6 - a_6a_1}.$$

Again, multiply (1) by a_3, (2) by $-a_1$, (3) by a_5, and add: as before, y will disappear, and we shall find the same value for x. Consequently two straight lines whose equations are

$$(1)\,a_6 - (2)\,a_4 = 0,$$

and

$$(1)\,a_3 - (2)\,a_1 = 0,$$

and which have a point in common, cut (3) in points whose abscissæ are equal, and which therefore coincide. Hence either two straight lines enclose a space, or (3) passes through the intersection of (1) and (2). Thus the existence of the point common to the three diagonals has been proved, and its abscissa found. To determine its ordinate, add (1), (2), (3), when x disappears, and we have

$$y = m\,\{a_1a_2(a_4+a_5) - a_2a_3(a_5+a_6) + a_3a_4(a_6+a_1) - a_4a_5(a_1+a_2) + a_5a_6(a_2+a_3) + a_6a_1(a_3+a_4)\},$$

divided by $\quad a_1a_2 - a_2a_3 + a_3a_4 - a_4a_5 + a_5a_6 - a_6a_1$.

If we call the coordinates of the point where the third and
[206] sixth sides of the hexagon meet $x_{\prime\prime\prime}$ $y_{\prime\prime\prime}$, and so of the other two points, these expressions for x and y become

$$x = \frac{x_{\prime\prime\prime}(x_4 - x_1) - x_{\prime\text{v}}(x_5 - x_2) + x_{\text{v}}(x_6 - x_3)}{x_1 - x_2 + x_3 - x_4 + x_5 - x_6},$$

$$y = \frac{x_1y_4 - x_2y_5 + x_3y_6 - x_4y_1 + x_5y_2 - x_6y_3}{x_1 - x_2 + x_3 - x_4 + x_5 - x_6}.$$

These expressions, as of course we should expect, are symmetrical.

In the last Number of this Journal a demonstration was given of a property of a parabola: That the circle which passes through the intersections of three tangents also passes through the focus. Although six demonstrations of this theorem have already appeared, yet the following is so simple that its insertion here may not be inappropriate.

Referring the parabola to the focus as origin, we can put the equation to the tangent under the form

$$y - \frac{x}{m} = a\left(m + \frac{1}{m}\right),$$

where a is one-fourth of the parameter, and m the trigonometrical tangent of the angle which the tangent makes with the axis of y. Hence, if x_1, y_1 be the coordinates of the point of intersection of

$$y - \frac{x}{m} = a\left(m + \frac{1}{m}\right),$$

with $$y - \frac{x}{m'} = a\left(m' + \frac{1}{m'}\right),$$

we have $$x_1 = a\,(mm' - 1),$$

$$y_1 = a\,(m + m'),$$

or putting $$m = \frac{\sin\alpha}{\cos\alpha},\quad m' = \frac{\sin\alpha'}{\cos\alpha'},$$

$$x_1 = a\,\frac{\cos(\alpha + \alpha')}{\cos\alpha\cos\alpha'},$$

$$y_1 = a\cdot\frac{\sin(\alpha + \alpha')}{\cos\alpha\cos\alpha'}.$$

To simplify these expressions turn the axes through an angle $= -(\alpha + \alpha' + \alpha'')$, and if x'', y'' be the new values of the coordinates, we find, after some simple reductions,

$$x'' = \frac{a\cos\alpha''}{\cos\alpha\cos\alpha'},\quad y'' = -\frac{a\sin\alpha''}{\cos\alpha\cos\alpha'}.$$

Squaring these and adding,

$$x''^2 + y''^2 = \frac{a^2}{\cos^2\alpha\cos^2\alpha'} = \frac{a}{\cos\alpha\cos\alpha'\cos\alpha''}\cdot\frac{a\cos\alpha''}{\cos\alpha\cos\alpha'},$$

or $$x''^2 + y''^2 = \frac{ax''}{\cos\alpha\cos\alpha'\cos\alpha''}.$$

Now this being symmetrical between $\alpha, \alpha', \alpha''$, will hold [207]
equally true of the three points of intersection, and it is the equation to a circle passing through the origin which is the focus, whose diameter coincides with the axis of x, and whose radius is

$$\frac{a}{2\cos\alpha\cos\alpha'\cos\alpha''}.$$

The chief advantage of this method besides its simplicity is, that it gives us very readily the radius of the circle, and the position of the diameter which passes through the focus.

It is easily seen that the distances from the focus of the three points of intersection of the tangents are respectively

$$r'' = \frac{a}{\cos\alpha\cos\alpha'},\quad r' = \frac{a}{\cos\alpha\cos\alpha''},\quad r = \frac{a}{\cos\alpha'\cos\alpha''}.$$

The area of the triangle formed by the intersection of the tangents, can be expressed by an elegant symmetrical function of $\tan\alpha$, $\tan\alpha'$, $\tan\alpha''$, that is, of m, m', m''. Since the lines joining the origin with the vertices of the triangle make angles $\alpha, \alpha', \alpha''$ with the diameter of the circle or the axis of x,

the angles they make with each other are $\alpha' - \alpha$, $\alpha'' - \alpha'$, $\alpha'' - \alpha$, and the area of the triangle will be

$$\tfrac{1}{2} rr' \sin(\alpha' - \alpha) + \tfrac{1}{2} r'r'' \sin(\alpha'' - \alpha') - \tfrac{1}{2} rr'' \sin(\alpha'' - \alpha).$$

Substituting for r, r', and r'' their values, this becomes

$$\frac{a^2}{2}\left\{\frac{\sin(\alpha' - \alpha)}{\cos^2\alpha'' \cos\alpha \cos\alpha'} + \frac{\sin(\alpha'' - \alpha')}{\cos^2\alpha \cos\alpha' \cos\alpha''} - \frac{\sin(\alpha'' - \alpha)}{\cos^2\alpha' \cos\alpha \cos\alpha''}\right\}.$$

Expanding the sines and making obvious reductions, we get

$$\frac{a^2}{2}\left\{\frac{\tan\alpha' - \tan\alpha}{\cos^2\alpha''} + \frac{\tan\alpha'' - \tan\alpha'}{\cos^2\alpha} + \frac{\tan\alpha - \tan\alpha''}{\cos^2\alpha'}\right\};$$

or grouping differently, and putting $\sec^2\alpha$ for $\dfrac{1}{\cos^2\alpha}$, and so on,

$$\frac{a^2}{2}\{\tan\alpha(\sec^2\alpha' - \sec^2\alpha'') + \tan\alpha'(\sec^2\alpha'' - \sec^2\alpha) + \tan\alpha''(\sec^2\alpha - \sec^2\alpha')\}.$$

Lastly, putting $1 + \tan^2\alpha = 1 + m^2$ for $\sec^2\alpha$, and so on, we find the area of the triangle to be

$$\frac{a^2}{2}\{m(m'^2 - m''^2) + m'(m''^2 - m^2) + m''(m^2 - m'^2)\},$$

which is quite symmetrical with respect to m, m', m''.

It will be easily seen, that the sides of the triangle are respectively

$$a\frac{m'' - m'}{\cos\alpha},\quad a\frac{m - m''}{\cos\alpha'},\quad a\frac{m' - m}{\cos\alpha''}.$$

[208] If these be called p, p', p'', and if ρ be the radius of the circle, by reduction, we obtain

$$p = 2\rho\sin(\alpha'' - \alpha'),\quad p' = 2\rho\sin(\alpha - \alpha''),\quad p'' = 2\rho\sin(\alpha' - \alpha).$$

If the values of the sines derived from these equations be substituted in the first expression for the area, it becomes

$$\frac{a}{2}\left(\frac{p}{\cos\alpha} + \frac{p'}{\cos\alpha'} + \frac{p''}{\cos\alpha''}\right)$$

R. L. E.

INVESTIGATION OF THE GENERAL TERM OF THE EXPANSION OF THE TRUE ANOMALY IN TERMS OF THE MEAN.

THE equations, by means of which the true anomaly θ is to be determined in terms of the mean nt, are

$$nt = u - e\sin u,$$

$$\tan \frac{\theta - \omega}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}.$$

Put $nt = z$, and let θ' be the value of θ when z is put for u; then

$$u = z + e \sin u,$$

and, by Lagrange's theorem,

$$\theta - \omega = \theta' - \omega + \frac{e}{1} \sin z \frac{d\theta'}{dz} + \frac{e^2}{1.2} \frac{d}{dz} \left\{(\sin z)^2 \frac{d\theta'}{dz}\right\} + \dots$$

$$= \Sigma \frac{e^p}{1.2.\dots.p} \frac{d^{p-1}}{dz^{p-1}} \left\{(\sin z)^p \frac{d\theta'}{dz}\right\},$$

p being taken from 0 to ∞.

Now $$\theta' - \omega = 2 \tan^{-1} \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{z}{2}\right);$$

$$\therefore \frac{d\theta'}{dz} = \frac{\sqrt{\frac{1+e}{1-e}} \left(\sec \frac{z}{2}\right)^2}{1 + \frac{1+e}{1-e} \left(\tan \frac{z}{2}\right)^2} = \frac{\sqrt{(1-e^2)}}{1 - e \cos z},$$

which may be expanded in the series

$$1 + 2\lambda \cos z + 2\lambda^2 \cos 2z + 2\lambda^3 \cos 3z + \dots.$$

where $$\lambda = \frac{1 - \sqrt{(1-e^2)}}{e} = \frac{e}{2} + \frac{e^3}{8} + \frac{e^5}{16} + \dots.$$

$$\therefore \frac{d\theta'}{dz} = 2\Sigma \lambda^m \cos mz,$$

if the term corresponding to $m = 0$ be divided by 2.

A general expression for $(\sin z)^p$ is required, in terms of sines or cosines of multiples of z. Assume [209]

$$\cos z + \sqrt{(-1)} \sin z = x, \quad \cos z - \sqrt{(-1)} \sin z = y,$$

then

$$\{2\sqrt{(-1)} \sin z\}^p = (x - y)^p = x^p - \frac{p}{1} x^{p-1} y + \frac{p(p-1)}{1.2} x^{p-2} y^2 - \dots$$

but $xy = 1$; therefore $x^{p-1}y = x^{p-2}$, $x^{p-2}y^2 = x^{p-4}$, &c.;

$$\therefore \{2\sqrt{(-1)} \sin z\}^p = x^p - \frac{p}{1} x^{p-2} + \frac{p(p-1)}{1.2} x^{p-4} - \dots$$

$$= \cos pz - \frac{p}{1} \cos (p-2) z + \frac{p(p-1)}{1.2} \cos (p-4) z - \dots$$

$$+ \sqrt{(-1)} \left\{\sin pz - \frac{p}{1} \sin (p-2) z + \frac{p(p-1)}{1.2} \sin (p-4) z - \dots\right\},$$

$$\text{and } \sqrt{(-1)} = \cos (4\mu + 1)\frac{\pi}{2} + \sqrt{(-1)} \sin (4\mu + 1)\frac{\pi}{2};$$

$$\therefore \{\sqrt{(-1)}\}^p = \cos (4\mu + 1) p \frac{\pi}{2} + \sqrt{(-1)} \sin (4\mu + 1) p \frac{\pi}{2}.$$

When p is an integer, $\{\sqrt{(-1)}\}^p$ will have only one value, therefore we may take $\mu = 0$;

$$\therefore \{\sqrt{(-1)}\}^p = \cos p \frac{\pi}{2} + \sqrt{(-1)} \sin p \frac{\pi}{2}.$$

$$\therefore (\sin z)^p = \frac{1}{2^p}\left\{\cos p\frac{\pi}{2} - \sqrt{(-1)} \sin p \frac{\pi}{2}\right\}\left[\cos pz - \frac{p}{1}\cos (p-2) z + \frac{p(p-1)}{1 \cdot 2}\cos (p-4) z - \dots + \sqrt{(-1)}\left\{\sin pz - \frac{p}{1}\sin (p-2) z + \frac{p(p-1)}{1 \cdot 2}\sin (p-4) z - \dots\right\}\right]$$

When p is an integer, $(\sin z)^p$ must be real; therefore taking only the real part of the second side,

$$(\sin z)^p = \frac{1}{2^p}\left[\cos\left(pz - p\frac{\pi}{2}\right) - \frac{p}{1}\cos\left\{(p-2) z - p\frac{\pi}{2}\right\} + \frac{p(p-1)}{1 \cdot 2}\cos\left\{(p-4) z - p\frac{\pi}{2}\right\} - \dots\right],$$

the series to be continued till it terminate of itself, which it will do, since p is an integer. We have therefore

$$(\sin z)^p = \frac{1}{2^p}\Sigma(-1)^q\frac{p(p-1)\dots(p-q+1)}{1 \cdot 2 \dots q}\cos\left\{(p-2q) z - p\frac{\pi}{2}\right\},$$

[210] where q is to be taken from 0. Hence

$$(\sin z)^p \frac{d\theta'}{dz}$$

$$= \frac{1}{2^p}\Sigma\Sigma(-1)^q \cdot \frac{p(p-1)\dots(p-q+1)}{1 \cdot 2 \dots q}\lambda^m \cdot 2\cos\left\{(p-2q) z - p\frac{\pi}{2}\right\}\cos mz$$

$$= \frac{1}{2^p}\Sigma\Sigma(-1)^q\frac{p(p-1)\dots(p-q+1)}{1 \cdot 2 \dots q}\lambda^m \times$$

$$\times\left[\cos\left\{(p-2q+m) z - p\frac{\pi}{2}\right\} + \cos\left\{(p-2q-m) z - p\frac{\pi}{2}\right\}\right].$$

Now in general

$$\frac{d^p \cos n\phi}{d\phi^p} = n^p \cos\left(n\phi + p\frac{\pi}{2}\right),$$

$$\therefore \frac{d^{p-1}}{dz^{p-1}}\left\{(\sin z)^p \frac{d\theta'}{dz}\right\}$$

$$= \frac{1}{2^p}\Sigma\Sigma\,(-1)^q \frac{p\,(p-1)\ldots(p-q+1)}{1\,.\,2\ldots q}\,\lambda^m \times$$

$$\times \{(p-2q+m)^{p-1}\sin(p-2q+m)\,z$$

$$+ (p-2q-m)^{p-1}\sin(p-2q-m)\,z\}.$$

Put $(p - 2q + m) = r$, and let us investigate the total coefficient of $r^{p-1}\sin rz$, when m and q vary from 0 upwards. Since $m = r - p + 2q$, when $q = 0$, $m = r - p$, so that the different values of

$$(-1)^q \frac{p(p-1)\ldots(p-q+1)}{1\,.\,2\ldots q}\,\lambda^m$$

give the series

$$\lambda^{r-p} + \frac{p}{1}\lambda^{r-p+2} + \frac{p\,(p-1)}{1\,.\,2}\lambda^{r-p+4}\ldots$$

$$= \lambda^{r-p}(1-\lambda^2)^p = \lambda^r(\lambda^{-1}-\lambda)^p,$$

in the case where r is greater than p. But if r be less than p, since m must not be negative, q must begin from $\frac{p-r}{2}$ or $\frac{p-r+1}{2}$, according as $p - r$ is even or odd, and m will begin from 0 or 1. The greatest value of q will be p, since for higher values the coefficient

$$\frac{p\,(p-1)\ldots\ldots(p-q+1)}{1.2\ldots\ldots q}$$

will vanish. The corresponding value of m is $r + p$, so that the coefficient is in this case, beginning with the greatest values of m and q,

$$(-1)^p\left\{\lambda^{r+p} - \frac{p}{1}\lambda^{r+p-2} + \frac{p\,(p-1)}{1\,.\,2}\lambda^{r+p-4} - \ldots\ldots\right\},$$

continued as long as the index of λ does not become negative. We may write it [211]

$$(-1)^p\lambda^{r+p}(1-\lambda^{-2})^p = \lambda^r(\lambda^{-1}-\lambda)^p,$$

if we observe that negative powers of λ are to be rejected. Also, the term independent of λ, when there is one, must be divided by 2, because it arises from the term corresponding to $m = 0$ in $\frac{d\theta'}{dz}$.

Next, put $p - 2q - m = r$, then $m = p - r - 2q$. Here p must be greater than r, otherwise m would be negative. We obtain, as before, the series

$$\lambda^{p-r} - \frac{p}{1}\lambda^{p-r-2} + \frac{p(p-1)}{1.2}\lambda^{p-r-4} - \ldots$$
$$= \lambda^{p-r}(1 - \lambda^{-2})^p = (-1)^p \lambda^{-r}(\lambda^{-1} - \lambda)^p,$$

with the same restriction as before. Hence, when $p > r$, we have for the multiplier of $r^{p-1}\sin rz$,

$$\{\lambda^r + (-1)^p \lambda^{-r}\}(\lambda^{-1} - \lambda)^p.$$

We have seen that when $p < r$, part only of this formula is required. But the same expression may be used in both cases, because when $p < r$, $\lambda^{-r}(\lambda^{-1} - \lambda)^p$ will contain only negative powers of λ, and is therefore to be rejected entirely. Hence the term in $\theta - \omega$, involving $\sin rnt$, is

$$\Sigma\,\frac{e^p}{1.2\ldots p}\,\frac{1}{2^p}\{\lambda^r + (-1)^p\lambda^{-r}\}(\lambda^{-1} - \lambda)^p r^{p-1}\sin rnt$$

$$= \left\{\lambda^r \Sigma \frac{\left(\frac{re}{2}\right)^p (\lambda^{-1} - \lambda)^p}{1.2\ldots p} + \lambda^{-r}\Sigma\frac{\left(-\frac{re}{2}\right)^p(\lambda^{-1} - \lambda)^p}{1.2\ldots p}\right\}\frac{\sin rnt}{r}$$

$$= \left\{\lambda^r \varepsilon^{\frac{re}{2}(\lambda^{-1} - \lambda)} + \lambda^{-r}\varepsilon^{-\frac{re}{2}(\lambda^{-1} - \lambda)}\right\}\frac{\sin rnt}{r}.$$

Here r may have any value, positive or negative. The terms arising from negative values of r are identical with those from equal positive values; and therefore if r have positive values only, including 0,

$$\theta - \omega = 2\Sigma\left\{\lambda^r \varepsilon^{\frac{re}{2}(\lambda^{-1} - \lambda)} + \lambda^{-r}\varepsilon^{-\frac{re}{2}(\lambda^{-1} - \lambda)}\right\}\frac{\sin rnt}{r}.$$

This expression must be applied only by developing it in the form in which it stands, dividing by 2 the terms in that development which do not involve λ, and rejecting all negative powers of that quantity.

S. S. G.

[212]

DEMONSTRATIONS OF THEOREMS IN THE DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

I PROPOSE in this Article to bring together the more important of the theorems in the Differential Calculus and in

the Calculus of Finite Differences, which, depending on one common principle, can be proved by the method of the separation of symbols. These theorems are usually demonstrated by induction in each particular case, which, although a method satisfactory so far as it goes, wants that generality which is desirable in Analytical demonstrations. As the ordinary Binomial Theorem is the basis on which these theorems are founded, it will be not amiss to say a few words by way of preface regarding the extent of its application, which being said once for all, will prevent useless repetition when we treat of each particular case.

The theorem that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \&c.$$

is originally proved when a and b are numbers, and $(a+b)^n$ represents the repetition of the operation n times, implying that n is an integer number. Having the form of the expansion once suggested, it can be shown, by the method of Euler, that the same form is true when n is a fraction or negative number; in which case the left-hand side of the equation acquires different meanings. Moreover, it will be found, on examining Euler's demonstration, that it includes not only these cases, but also all those in which a, b, and n are operations subject to certain laws; for it may be seen, that in the proof no other properties are presumed than that a, b, and n are distributive and commutative functions, and that a^n, b^n are subject to the laws of index functions. These laws are,

(1) The commutative, $ab = ba$,

(2) The distributive, $c(a+b) = ca + cb$,

(3) The index law, $a^m.(a^n) = a^{m+n}$.

Now, since it can be shown that the operations both in the Differential Calculus and the Calculus of Finite Differences are subject to these laws, the Binomial Theorem may be at once assumed as true with respect to them, so that it is not necessary to repeat the demonstration of it for each case.* This being premised, I proceed to consider the particular cases of the applications of these theorems.

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* It is scarcely necessary to add, that those theorems which depend on the binomial, as the polynomial and exponential, are equally extensive, so that they too may be applied to the Differential Calculus and Calculus of Finite Differences.

[213] 1. If $u = f(x, y)$ be a function of two independent variables, we find that

$$d(u) = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = \left(\frac{d}{dx}\,dx + \frac{d}{dy}\,dy\right)u,$$

by separating the symbols. Now, if we wish to find the n^{th} differential of a function of two variables, we have merely, by the principle of indices, to affix the index n to the sign of operation on both sides, when we get

$$d^n(u) = \left(\frac{d}{dx}\,dx + \frac{d}{dy}\,dy\right)^n u.$$

Now the operation on the second side, being a binomial raised to a power, may, by what has just been said, be expanded by the binomial theorem, so that we have

$$d^n u = \frac{d^n u}{dx^n}\,dx^n + n\,\frac{d^{n-1}u}{dx^{n-1}}\,\frac{du}{dy} + \frac{n(n-1)}{1.2}\,\frac{d^{n-2}u}{dx^{n-2}}\,\frac{d^2u}{dy^2} - \&\text{c}.$$

This theorem, which can be proved by induction only for positive integer powers of n, that is, for cases of ordinary differentiation, is shown by this method to be true when n is fractional or negative, that is, in the cases of integration and general differentiation.

If we suppose u to be a function of three or more variables, we might, by means of the polynomial theorem, expand $d^n.(u)$; but it is not necessary to dwell upon the result, as there is little interest attached to it.

2. I shall next proceed to the elegant theorem of Leibnitz, for finding the n^{th} differential of the product of two functions, a theorem which, when generalized, is most fertile in consequences.

Let u, v be the two functions. Then

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}.$$

This may be put under the form

$$\frac{d}{dx}(u\,v) = \left(\frac{d'}{dx} + \frac{d}{dx}\right)u\,v,$$

if we agree to represent by $\frac{d'}{dx}$ an operation which acts on v, but not on u, and by $\frac{d}{dx}$ an operation which acts on u and

not on v. These operations from their nature are distributive, and as they are independent of each other, they must be commutative; hence they come under the circumstances to which the binomial theorem applies. Taking then the n^{th} differential,

$$\left(\frac{d}{dx}\right)^n (u\,v) = \left(\frac{d'}{dx} + \frac{d}{dx}\right)^n u\,v$$

$$= \left\{\left(\frac{d'}{dx}\right)^n + n\left(\frac{d'}{dx}\right)^{n-1} \frac{d}{dx} + \frac{n.n-1}{1.2}\left(\frac{d'}{dx}\right)^{n-2}\left(\frac{d}{dx}\right)^2 + \text{\&c.}\right\} u\,v;$$

or applying the operations directly to the quantities [214] which they affect,

$$= u\,\frac{d^n v}{dx^n} + n\,\frac{du}{dx}\,\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{1.2}\,\frac{d^2u}{dx^2}\,\frac{d^{n-2}v}{dx^{n-2}} + \text{\&c.}$$

This theorem is true, like the former, when n is negative or fractional. In the former case, the form is the same as the series at which we arrive by integration by parts, which we thus see to be a particular case of the theorem of Leibnitz.

3. In this expression, when n is negative, let $v = 1$. Then

$$\frac{d^{-n}v}{dx^{-n}} = \frac{x^n}{n!},\quad \frac{d^{-(n+1)}v}{dx^{-(n+1)}} = \frac{x^{n+1}}{(n+1)!},\ \text{\&c.}$$

so that

$$\left(\frac{d}{dx}\right)^{-n} u = \int^n dx^n\, u = \frac{x^n}{n!}\,u - n\,\frac{x^{n+1}}{(n+1)!}\,\frac{du}{dx}$$
$$+ \frac{n(n+1)}{1.2}\,\frac{x^{n+2}}{(n+2)!}\,\frac{d^2u}{dx^2} + \text{\&c.}$$

$$= \frac{x^{n-1}}{(n-1)!}\left(\frac{x}{n}\,u - \frac{x^2}{n+1}\,\frac{du}{dx} + \frac{1}{1.2}\,\frac{x^3}{n+2}\,\frac{d^2u}{dx^2} - \text{\&c.}\right)$$

which is the general expression for the n^{th} integral of any function.

4. In this last formula, if we make $n = 1$ when it becomes a simple integral, we have

$$\int dx\, u = xu - \frac{x^2}{1.2}\,\frac{du}{dx} + \frac{x^3}{1.2.3}\,\frac{d^2u}{dx^2} - \text{\&c.},$$

the well-known series of Bernouilli; which thus appears to be also a particular case of the theorem of Leibnitz when extended to general indices.

5. In this theorem, let us suppose $v = \varepsilon^{ax}$; then as

$$\frac{dv}{dx} = a\varepsilon^{ax} = av,$$

we have $\frac{d'}{dx} = a$, and therefore

$$\frac{d^n}{dx^n}(\varepsilon^{ax}u) = \left(a + \frac{d}{dx}\right)^n u\varepsilon^{ax};$$

whence $$\left(a + \frac{d}{dx}\right)^n u = \varepsilon^{-ax}\left(\frac{d}{dx}\right)^n \varepsilon^{ax}u,$$

which is the theorem given in Art. v. of our first Number.

6. In the Calculus of Finite Differences there are more theorems than in the Differential Calculus depending on the expansion of a binomial, in consequence of the relation which subsists between two kinds of operations, that of taking the increment and that of taking the difference. It is not usual to use a separate symbol for the former, but in Art II. of the second Number of this Journal, I adopted the symbol D to represent this operation, as it simplified greatly the expressions. [215] For the same reason I shall continue to employ it, and I hope that its utility will compensate for any disadvantage which may accrue from using a new notation. Before proceeding, I will say a few words concerning the operation represented by this symbol D. Its definition is, that

$$Df(x) = f(x + 1).$$

Now we know by Taylor's theorem that

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x + h),$$

whatever h may be; making $h = 1$, we have

$$\varepsilon^{\frac{d}{dx}} = f(x + 1).$$

Consequently $$D = \varepsilon^{\frac{d}{dx}};$$

from which we see that $D^h f(x) = f(x + h)$, whatever h may be.

Also, since

$$\Delta f(x) = f(x + 1) - f(x) = Df(x) - f(x) = (D - 1)f(x),$$

we have $$\Delta = D - 1 \text{ and } D = 1 + \Delta.$$

Consequently, D being a linear compound of commutative and distributive operations, is also a commutative and distributive operation. It is also subject to the laws of index functions,

since $$D^k D^h f(x) = D^k f(x + h) = f(x + h + k) = D^{h+k} f(x).$$

7. This being premised, since we have

$$Du_x = (1 + \Delta)\, u_x,$$
$$D^n u_x = (1 + \Delta)^n\, u_x;$$

and by the binomial theorem,

$$D^n u_x = \left\{1 + n\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \frac{n(n-1)(n-2)}{1.2.3}\Delta^3 + \&\text{c.}\right\} u_x.$$

or
$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&\text{c.}$$

Again, $D^n u_x = (D^{-1})^{-n}\, u_x = \{D^{-1}(D - \Delta)\}^{-n}\, u_x$
$= (1 - \Delta D^{-1})^{-n}\, u_x;$

whence by the binomial theorem

$$D^n u_x = \left\{1 + n\Delta D^{-1} + \frac{n(n+1)}{1.2}\Delta^2 D^{-2} + \frac{n(n+1)(n+2)}{1.2.3}\Delta^3 D^{-3} + \&\text{c.}\right\} u_x,$$

and therefore

$$u_{x+n} = u_x + n\Delta u_{x-1} + \frac{n(n+1)}{1.2}\Delta u_{x-2} + \&\text{c.}$$

8. Besides these there are two other expressions for [216]
u_{x+n}, which, though not depending on the binomial theorem, are founded on the same principles:

$$u_{x+n} - u_x = (D^n - 1)\, u_x$$
$$= \Delta\,(D-1)^{-1}(D^n - 1)\, u_x,$$

since $\Delta = D - 1$.

Expanding in the same way as we would expand $\dfrac{x^n - 1}{x - 1}$, we find

$$u_{x+n} - u_x = \Delta\,(1 + D + D^2 + \&\text{c.} + D^{n-1})\, u_x,$$

and therefore $u_{x+n} = u_x + \Delta u_x + \Delta u_{x+1} + \Delta u_{x+2} + \&\text{c.} + \Delta u_{x+n-1}$.

Similarly, $u_{x+n} - \Delta^n u_x = (D^n - \Delta^n)\, u_x = (D - \Delta)^{-1}(D^n - \Delta^n)\, u_x$,

since $$D - \Delta = 1.$$

Therefore, expanding as we would expand $\dfrac{x^n - a^n}{x - a}$, we find

$$u_{x+n} - \Delta^n u_x = (D^{n-1} + \Delta D^{n-2} + \Delta^2 D^{n-3} + \&\text{c.} + \Delta^{n-2} D + \Delta^{n-1})\, u_x,$$

and therefore

$$u_{x+n} = u_{x+n-1} + \Delta u_{x+n-2} + \&\text{c.} + \Delta^{n-2} u_{x+1} + \Delta^{n-1} u_x + \Delta^n u_x.$$

9. Corresponding to these theorems for $D^n u_x$, we have theorems for $\Delta^n u_x$,

Since $\Delta u_x = (D-1)\,u_x$, $\Delta^n u_x = (D-1)^n\,u_x$,

and therefore by the binomial theorem

$$\Delta^n u_x = \left\{D^n - nD^{n-1} + \frac{n\,(n-1)}{1.2}\,D^{n-2} - \&c.\right\} u_x$$

$$= u_{x+n} - nu_{x+n-1} + \frac{n\,(n-1)}{1.2}\,u_{x+n-2} - \&c.$$

Again, $\Delta^r u_{x+n} = \Delta^r D^r u_{x+n-r} = \Delta^r \left\{D^{-1}(D-\Delta)\right\}^{-r} u_{x+n-r}$,

$$= \Delta^r (1 - \Delta D^{-1})^{-r}\, u_{x+n-r},$$

and expanding

$$\Delta^r u_{x+n} = \Delta^r \left\{1 + r\Delta D^{-1} + \frac{r\,(r+1)}{1.2}\,\Delta^2 D^{-2} + \dots\right\} u_{x+n-r}$$

$$= \Delta^r u_{x+n-r} + r\Delta^{r+1} u_{x+n-r-1}$$

$$+ \frac{r\,(r+1)}{1.2}\,\Delta^{r+2} u_{x+n-r-2} - \dots$$

10. Connected with this subject is a formula for the transformation of series, which is useful for the purpose of changing diverging into converging series. The proof of this, which is usually made to depend on the theory of generating functions, can be much more simply derived from the theory I am here [217] developing, and the same may be said of all theorems usually demonstrated by generating functions. Let the given series be

$$S = y_x + y_{x+1} + y_{x+2} + y_{x+3} + \dots.$$

$$= (1 + D + D^2 + D^3 + \dots.)\,y_x = (1-D)^{-1}\,y_x;$$

and let it be desired to change this into one depending on

$$ay_x + a_1 y_{x+1} + a_2 y_{x+2} + \dots. + a_n y_{x+n}$$

$$= (a + a_1 D + a_2 D^2 + \dots. + a_n D^n)\,y_x = \Delta y_x,$$

if we put $\nabla = a + a_1 D + a_2 D^2 + \dots. + a_n D^n$.

Now it is to be observed, that any algebraic combination of the symbols D and Δ with constants will be likewise subject to the same laws of combination as these symbols, and may therefore be treated in the same way.

Hence, making $a + a_1 + a_2 + \dots. + a_n = K$, we shall have

$$S = (1-D)^{-1}\,y_x = (K - \nabla)^{-1}\,(K - \nabla)\,(1-D)^{-1} y_x,$$

since the operations $(K - \nabla)^{-1}$, $K - \nabla$ destroying each other, do not affect the equation. Now

$$(1 - D)^{-1}(K - \nabla) = (1 - D)^{-1}\{a_1(1 - D) + a_2(1 - D^2) + a_3(1 - D^3) + \ldots + a_n(1 - D^n)$$

$$= a_1 + a_2(1 + D) + a_3(1 + D + D^2) + \ldots + a_n(1 + D + D^2 + \ldots + D^{n-1})$$

$$= a_1 + a_2 + a_3 + \ldots + a_n + (a_2 + a_3 + \ldots + a_n)D + (a_3 + \ldots + a_n)D^2 + \ldots + \ldots$$

And therefore

$$S = (a_1 + a_2 + a_3 + \ldots + a_n)(K - \nabla)^{-1} y_x + (a_2 + a_3 + \ldots + a_n)(K - \nabla)^{-1} y_{x+1} + (a_3 + \ldots + a_n)(K - \nabla)^{-1} y_{x+2} + \ldots + \ldots$$

and expanding $(K - \nabla)^{-1}$, we find

$$S = (a_1 + a_2 + a_3 + \ldots + a_n)\left(\frac{y_x}{K} + \frac{\nabla y_x}{K^2} + \frac{\nabla^2 y_x}{K^3} + \ldots\right)$$

$$+ (a_2 + a_3 + \ldots + a_n)\left(\frac{y_{x+1}}{K} + \frac{\nabla y_{x+1}}{K^2} + \frac{\nabla^2 y_{x+1}}{K^3} + \ldots\right)$$

$$+ (a_3 + \ldots + a_n)\left(\frac{y_{x+2}}{K} + \frac{\nabla y_{x+2}}{K^2} + \frac{\nabla^2 y_{x+2}}{K^3} + \ldots\right) + \ldots$$

which is the required transformation for S.

11. In the particular case where

$$S = ax + a_1x^2 + a_2x^3 + a_3x^4 + \ldots,$$

Euler has employed a very elegant method of transfor- [218] mation, the reason for which appears very clearly, if we follow the method of the separation of symbols.

Let us suppose $a, a_1, a_2, a_3, \ldots$ to be terms of a series which can be derived one from the other by a certain law, so that $a_1 = Da$, $a_2 = D^2a$, and so on. Then

$$S = (x + x^2D + x^3D^2 + \ldots)\,a$$

$$= x(1 - xD)^{-1}a = x(1 - x - x\Delta)^{-1}a$$

$$= \frac{x}{1 - x}\left(1 - \frac{x}{1 - x}\Delta\right)^{-1} a$$

$$= \frac{x}{1 - x}\left\{1 + \frac{x}{1 - x}\Delta + \frac{x^2}{(1 - x)^2}\Delta^2 + \ldots\right\} a$$

$$= \frac{ax}{1 - x} + \Delta a\frac{x^2}{(1 - x)^2} + \Delta^2 a\frac{x^3}{(1 - x)^3} + \ldots$$

12. The expression for the total difference of a function of two variables, in terms of the partial differences, is not so simple as its analogue in the Differential Calculus. If we represent the total difference by Δ, and the partial differences with respect to x and y by Δ_x, Δ_y, and the corresponding increments by D_x, D_y, we have, since

$$\Delta_x u_{x,y+1} = u_{x+1,y+1} - u_{x,y+1},$$

and

$$\Delta_y u_{x+1,y} = u_{x+1,y+1} - u_{x+1,y},$$

by adding them together, and subtracting from

$$2\Delta u_{x,y} = 2\,(u_{x+1,y+1} - u_{x,y}),$$

$$2\Delta u_{x,y} - \Delta_x u_{x,y+1} - \Delta_y u_{y,x+1} = u_{x,y+1} - u_{x,y} + u_{x+1,y} - u_{x,y},$$

whence $\quad 2\Delta u_{x,y} = \Delta_y D_x u_{x,y} + \Delta_x D_y u_{x,y} + \Delta_y u_{x,y} + \Delta_x u_{x,y},$

or $\quad \Delta u_{x,y} = \frac{1}{2}\left\{\Delta_y (1 + D_x) + \Delta_x (1 + D)\right\} u_{x,y}.$

Now, since all the symbols are relatively commutative, inasmuch as $D.\Delta u_x = \Delta . Du_x$ when they refer to the same variable, and as when referring to different variables they are wholly independent, and therefore commutative; and since all the symbols are also distributive, the binomial theorem may be here applied, and therefore

$$\begin{aligned}\Delta^n u_{x,y} &= \frac{1}{2^n}\left\{\Delta_y (1 + D_x) + \Delta_x (1 + D_y)\right\}^n u_{x,y} \\ &= \frac{1}{2^n}\left\{\Delta_y^n (1 + D_x)^n + n\Delta_y^{n-1} (1 + D_x)^{n-1} \Delta_x (1 + D_y)\right. \\ &\quad \left. + \frac{n(n-1)}{1.2} \Delta_y^{n-2} (1 + D_x)^{n-2} \Delta_x^2 (1 + D_y)^3 + \dots\right\} u_{x,y}.\end{aligned}$$

If each term be expanded, and the operations indicated by D_x, D_y be effected, we shall obtain a result in Δ_x and Δ_y; but it is so complicated, that it is better to keep it in the unexpanded form, as we thus see the law of formation more distinctly.

13. It is also obvious, that as

$$\Delta u_{x,y} = (D - 1)\, u_{x,y},$$

when Δ and D are total operations referring to both variables,

$$\begin{aligned}\Delta^n u_{x,y} &= \left(D^n - n\, D^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} - \dots\right) u_{x,y} \\ &= u_{x+n,y+n} - nu_{x+n-1,y+n-1} + \dots\end{aligned}$$

and similarly

$$D^n u_{x,y} = u_{x,y} + n\Delta u_{x,y} + \frac{n(n-1)}{1.2} \Delta^2 u_{x,y} + \dots .$$

14. We shall proceed now to the operations on the products of two or more functions of the same variable.

$$\Delta u_x v_x = u_{x+1}\, v_{x+1} - u_x v_x = (DD_1 - 1)\, u_x v_x,$$

where we suppose D to refer to u_x and D_1 to v_x. Therefore
$\Delta^n u_x v_x = (DD_1 - 1)^n\, u_x v_x$

$$= \left(D^n D_1^n - n\, D^{n-1} D_1^{n-1} + \frac{n\,(n-1)}{1.2}\, D^{n-2}\, D_1^{n-2} - \ldots\right) u_x v_x;$$

therefore $\qquad \Delta^n u_x v_x = u_{x+n} v_{x+n} - n u_{x+n-1} v_{x+n-1} + \ldots$

This is true whatever n is. Let it be negative, then

$$\Delta^{-n} u_x v_x = \Sigma^n u_x v_x = u_{x-n} v_{x-n} + n u_{x-n-1} v_{x-n-1} + \ldots$$

15. The n^{th} difference of the product of two functions may be expanded in another manner. Since

$$\begin{aligned}\Delta u_x v_x &= u_{x+1} v_{x+1} - u_x v_x \\ &= u_x \Delta v_x + v_{x+1} \Delta u_x \\ &= (\Delta + D\Delta_1)\, u_x v_x,\end{aligned}$$

(where Δ, D refer to u_x and Δ_1 to v_x),

$\Delta^n u_x v_x = (\Delta + D\Delta_1)^n u_x v_x$

$$= \left(\Delta^n + \Delta^{n-1}\Delta_1 D + \frac{n\,(n-1)}{1.2}\, \Delta^{n-2}\, \Delta_1^{\,2} D^2 + \ldots\right) u_x v_x;$$

therefore

$$\Delta^n u_x v_x = v_x \Delta^n u_x + n \Delta v_x \Delta^{n-1}\, u_{x+1} + \frac{n\,(n-1)}{1.2}\, \Delta^2 v_x \Delta^{n-2}\, u_{x+2} + \ldots$$

When n becomes negative, $\Delta^{-n} u_x v_x = \Sigma^n u_x v_x$, and therefore

$$\Sigma^n\,(u_x v_x) = v_x\, \Sigma^n u_x - n \Delta v_x \Sigma^{n+1} u_{x+1} + \frac{n\,(n+1)}{1.2}\, \Delta^2 v_x \Sigma^{n+2} u_{x+2} - \ldots$$

which is the formula for integration by parts; and when $n = 1$,

$$\Sigma\,(u_x v_x) = v_x \Sigma u_x - \Delta v_x \Sigma^2 u_{x+1} + \Delta^2 v_x \Sigma^3 u_{x+2} - \ldots$$

16. If we suppose $u_x = x^0 = 1$ in the former expression, we have, since $u_x = u_{x+1} = u_{x+2}$, &c.,

$$\Sigma^n u_x = \Sigma^n \,.\, 1 = \frac{x\,(x+1) \quad (x+n-1)}{(n)!},$$

and therefore [220]

$$\Sigma^n\,(v_x) = \frac{x\,(x+1)\ldots(x+n-1)}{(n-1)!} \left\{\frac{v_x}{n} - \frac{(x+n)}{1}\,\frac{\Delta v_x}{n+1} + \frac{(x+n)\,(x+n+1)}{1.2}\,\frac{\Delta^2 v_x}{n+2} - \ldots\right\},$$

which, when $n = 1$, becomes

$$\Sigma (v_x) = xv_x - \frac{x(x+1)}{1.2} \Delta v_x + \frac{x(x+1)(x+2)}{1.2.3} \Delta^2 v_x - \dots,$$

a series which bears a close analogy with that of Bernouilli in the Integral Calculus.

17. Again, since

$$\Delta^n u_x v_x = (DD_1 - 1)^n u_x v_x,$$

if we make $v_x = a^x$, $D_1 a^x = a^{x+1} = a \,.\, a^x$, so that $D_1 = a$, and

$$\Delta^n u_x a^x = (Da - 1)^n u_x a^x,$$

and $$(Da - 1)^n u_x = a^{-x} \,.\, \Delta^n (u_x a^x).$$

If for a we put $\frac{1}{a}$, we obtain

$$(D - a)^n u_x = a^{x+n} \Delta_n (u_x a^{-x}),$$

which is the theorem given in Article II. of the second Number of this Journal.

18. The connexion which exists between the Differential Calculus and the Calculus of Finite Differences, gives rise to various elegant theorems; the first of which is the celebrated theorem of Lagrange, that

$$\Delta^n u_x = (\epsilon^{\frac{d}{dx}} - 1)^n u_x.$$

For as we have

$$\Delta u_x = (D - 1)\, u_x = (\epsilon^{\frac{d}{dx}} - 1)\, u_x,$$

raising the symbol of operation to the n^{th} power on each side,

$$\Delta^n u_x = (\epsilon^{\frac{d}{dx}} - 1)^n u_x.$$

It is usual to make the proof of this theorem a matter of some difficulty, but it follows at once from the theory of the laws of combination of the symbols. It is true whatever n may be, and therefore when n is negative, or

$$\Sigma^n u_x = (\epsilon^{\frac{d}{dx}} - 1)^{-n} u_x,$$

or for the particular value 1 of n,

$$\Sigma u_x = (\epsilon^{\frac{d}{dx}} - 1)^{-1} u_x.$$

The second side, when expanded by the numbers of Bernouilli, gives

$$\Sigma u_x = \left\{\left(\frac{d}{dx}\right)^{-1} - \frac{1}{2} + \frac{B_1}{1.2}\frac{d}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3}{dx^3} + \ldots\right\} u_x \qquad [221]$$

$$= \int u_x dx - \frac{u_x}{2} + \frac{B_1}{1.2}\frac{du_x}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3u_x}{dx^3} + \ldots$$

19. The theorem for expressing the n^{th} difference in terms of the n^{th} and higher differential coefficients, may be derived very readily without expansion from the fundamental theorem

$$\Delta u_x = (\varepsilon^{\frac{d}{dx}} - 1)\, u_x;$$

for we shall also have

$$\Delta\,(\varepsilon^{h\frac{d}{dx}} u) = \varepsilon^{h\frac{d}{dx}}(\varepsilon^{\frac{d}{dx}} - 1)\, u_x, \quad \text{when } h = 0.$$

But $\varepsilon^{h\frac{d}{dx}}(\varepsilon^{\frac{d}{dx}} - 1)$ is the difference of $\varepsilon^{h\frac{d}{dx}}$, taken with respect to h, and may be represented by $\Delta_h\,\varepsilon^{h\frac{d}{dx}}$, where Δ_h implies that the sign of operation affects h only. Hence we have

$$\Delta\,(\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h\,(\varepsilon^{h\frac{d}{dx}} u_x), \quad \text{when } h = 0.$$

By this artifice the symbol of operation is transferred from the x to the h. Now, taking the n^{th} difference on both sides,

$$\Delta^n\,(\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h^{\,n}\,(\varepsilon^{h\frac{d}{dx}} u_x), \quad \text{when } h = 0.$$

On expanding $\varepsilon^{h\frac{d}{dx}}$ on the second side, and effecting the operation $\Delta_h^{\,n}$, it appears that all the terms will vanish till the $(n+1)^{\text{th}}$, so that replacing h by 0 we have the usual formula

$$\Delta^n u_x = \frac{\Delta^n 0^n}{n!}\frac{d^n u}{dx^n} + \frac{\Delta^n 0^{n+1}}{(n+1)!}\frac{d^{n+1}u}{dx^{n+1}} + \ldots$$

20. The same method affords an easy proof of a theorem first given by Sir John Herschel in the *Phil. Trans.*, 1816, for expanding any function of ε^t.

$$f(\varepsilon^t) = f(\varepsilon^t)\,\varepsilon^{xt} \text{ when } x = 0.$$

And since

$$\varepsilon^{\frac{d}{dx}}\,.\,\varepsilon^{xt} = \varepsilon^t \text{ when } x = 0,$$

$$f(\varepsilon^t) = f(\varepsilon^{\frac{d}{dx}})\,\varepsilon^{xt} \text{ when } x = 0,$$

or expanding ε^{xt}, putting 0 for x and $1 + \Delta$ for $\varepsilon^{\frac{d}{dx}}$, we get

$$f(\varepsilon^t) = f(1+\Delta)\,1 + f(1+\Delta)\,0\,.\,t + f(1+\Delta)\,\frac{0^2t^2}{1.2} + \ldots$$

which is the form given by Sir John Herschel.

R 2

I cannot mention the name of this mathematician without correcting an error into which I fell in Article v. of the first [222] Number of this Journal. I there stated that, so far as I knew, Brisson was the first person who had applied the method of the separation of symbols to the solution of differential equations. I have since found that Sir John Herschel was really the first person who did so, in a paper published in the *Philosophical Transactions* for 1816, five years before the date of Brisson's memoir. It is much to be regretted, that neither Sir John Herschel himself, nor any other person, followed up this method, which is calculated to be of so much use in the higher analysis. Perhaps this may have arisen from the theory of the method not having been properly laid down, so that a certain degree of doubt existed as to the correctness of the principle. I trust, however, that the various developments which I have given in several articles in this Journal, of the principles of the method as well as the proofs of its utility, are sufficient for removing all doubts on this head, and that it will now be regarded as a powerful instrument in the hands of mathematicians.

D. F. G.

ON THE CONDITIONS OF EQUILIBRIUM OF A RIGID SYSTEM, &c.*

SUPPOSE the forces $P, P', P'', \ldots$ applied at the points (x, y, z), (x', y', z'), $(x'', y'', z''), \ldots$ of any rigid system, and let α, β, γ, α', β', γ', .. be the angles which their directions make with the axes of coordinates.

Let AO be a line drawn from the origin A to any point O taken arbitrarily, and through O let a plane be drawn perpendicular to AO.

The effect of the forces will not be altered by removing their points of application to the points where their directions meet this plane. This being done, we may resolve each force into two; one in the plane, and coinciding in magnitude and direction with the projection of the force upon the plane, and the other perpendicular to it. We shall thus have two sets of forces; one set lying entirely in the assumed plane, and the other perpendicular to it, and parallel to each other; and it is plain that these two sets must be separately in equilibrium.

* From a Correspondent.

Now a condition of equilibrium for parallel forces is, that their sum shall equal nothing. And a condition for forces in one plane is, that the sum of their moments about any point in it shall equal nothing.

To express these two conditions, let θ be the angle [223] between the direction of the force P, and the line AO; then the two resolved parts of P will be $P \cos \theta$ perpendicular to the assumed plane, and $P \sin \theta$ in the plane. Also let q be the perpendicular drawn from O upon the direction of the latter force, *i. e.* upon the projection of P on the plane. And calling θ', q', . . the corresponding quantities for P', P'', . . the two conditions above mentioned will be

$$(1)\quad P \cos \theta + P' \cos \theta' + \ldots = 0,$$

$$(2)\quad Pq \sin \theta + P'q' \sin \theta' + \ldots = 0.$$

Let a, b, c be the cosines of the angles between the line AO and the three axes respectively, then

$$(3)\ \cos \theta = a \cos \alpha + b \cos \beta + c \cos \gamma,$$

$$(4)\ \sin^2 \theta = (b \cos \gamma - c \cos \beta)^2 \times (c \cos \alpha - a \cos \gamma)^2 + (a \cos \beta - b \cos \alpha)^2.$$

Hence, substituting for $\cos \theta$, $\cos \theta'$, ... in (1), we have

$$a\,(P \cos \alpha + P' \cos \alpha' + \ldots) + b\,(P \cos \beta + P' \cos \beta' + \ldots) + c\,(P \cos \gamma + P' \cos \gamma' + \ldots) = 0,$$

or

$$aX + bY + cZ = 0,$$

and therefore, since any two of the cosines a, b, c are arbitrary, we must have separately

$$X = 0, \quad Y = 0, \quad Z = 0.$$

To express the condition (2). Let a plane be drawn containing the direction of the force P, and perpendicular to the assumed plane upon which we have projected the forces. The intersection of these two planes will evidently be the projection of P upon the latter, and a perpendicular drawn from the origin upon the former will be precisely equal to the perpendicular q. Suppose l, m, n are the cosines of the angles between this perpendicular and the three axes, then the equation to the plane in question will be

$$(5)\quad l\,(\xi - x) + m\,(\eta - y) + n\,(\zeta - z) = 0,$$

and therefore evidently

$$(6)\quad q = lx + my + nz.$$

Also l, m, n will be subject to the equations expressing the two conditions that the plane (5) contains the force P, and

is perpendicular to the plane on which the forces are projected; these are evidently

$$l \cos \alpha + m \cos \beta + n \cos \gamma = 0$$
$$la + mb + nc = 0,$$

from which we deduce immediately

$$\frac{l}{b \cos \gamma - c \cos \beta} = \frac{m}{c \cos \alpha - a \cos \gamma} = \frac{n}{a \cos \beta - b \cos \alpha} = \frac{1}{\sin \theta}$$

(see equation (4); and the theorem at page 187).

[224] Hence, substituting for l, m, n in (6), and multiplying by P, we obtain

$$Pq \sin \theta = P(b \cos \gamma - c \cos \beta)x + P(c \cos \alpha - a \cos \gamma)y + P(a \cos \beta - b \cos \alpha)z.$$

This is the moment of the projection of P upon the arbitrarily assumed plane, with reference to the point O. If we form similar equations for the other forces and collect their sum, we find

$$Pq \sin \theta + P'q' \sin \theta' + \ldots = a.\{P(z \cos \beta - y \cos \gamma) + P'(z' \cos \beta' - y' \cos \gamma') + \}\ldots$$
$$+ b.\{P(x \cos \gamma - z \cos \alpha) + \ldots\} + c.\{P(y \cos \alpha - x \cos \beta) + \ldots\}.$$

The expression on the right of this equation, which we may write for shortness $aL + bM + cN$, must equal nothing in the case of equilibrium. Hence, as before, we must have separately $L = 0, \quad M = 0, \quad N = 0.$

These equations, together with the three $X = 0$, $Y = 0$, $Z = 0$, are therefore necessary conditions of equilibrium, and it is easily seen that they are sufficient.

It appears from the preceding demonstration that the moment of the forces about any axis passing through the origin, and making with the axes of coordinates angles whose cosines are a, b, c, is

$$(7) \quad aL + bM + cN.$$

To find the *principal moment*, we must make this expression a maximum.

Putting therefore $Lda + Mdb + Ndc = 0$, and combining this with the equation $ada + bdb + cdc = 0$, we find (since any two of the differentials are independent)

$$(8) \; \frac{L}{a} = \frac{M}{b} = \frac{N}{c} = \sqrt{(L^2 + M^2 + N^2)} = aL + bM + cN \text{ (see p. 187).}$$

This determines the value of the principal moment, viz.

$$\sqrt{(L^2 + M^2 + N^2)},$$

and also the values of a, b, c, which give the position of its axis.

If through any point (ξ, η, ζ) we draw three lines parallel to the axes of coordinates, the moments round these lines, which we may call L', M', N', will evidently be found by putting $x - \xi$, $y - \eta$, $z - \zeta$ for x, y, z, in the values of L, M, N. This substitution gives

$$L' = L + Z\eta - Y\zeta,$$
$$M' = M + X\zeta - Z\xi,$$
$$N' = N + Y\xi - X\eta.$$

The principal moment with reference to this point will be

$$\sqrt{(L'^2 + M'^2 + N'^2)};$$

and if we investigate the conditions which make this a *minimum*, we find (equating to 0 the partial differential [225] coefficients with regard to ξ, η, ζ) equations which may be written as follows:

$$(9)\quad \frac{L + Z\eta - Y\zeta}{X} = \frac{M + X\zeta - Z\xi}{Y} = \frac{N + Y\xi - X\eta}{Z};$$

that is to say,

$$(10)\quad \frac{L'}{X} = \frac{M'}{Y} = \frac{N'}{Z} = \frac{\sqrt{(L'^2 + M'^2 + N'^2)}}{\sqrt{(X^2 + Y^2 + Z^2)}} = \frac{L'X + M'Y + N'Z}{X^2 + Y^2 + Z^2},$$

and therefore, (observing that $L'X + M'Y + N'Z = LX + MY + NZ$,)

$$\sqrt{(L'^2 + M'^2 + N'^2)} = \frac{LX + MY + NZ}{\sqrt{(X^2 + Y^2 + Z^2)}},$$

which determines the least principal moment.

If we equate two and two the expressions (9), we obtain by an easy transformation, (putting $X^2 + Y^2 + Z^2 = R^2$),

$$R^2\xi + NY - MZ = X\,(X\xi + Y\eta + Z\zeta),$$

and similar expressions for η, ζ; whence

$$\frac{R^2\xi + NY - MZ}{X} = \frac{R^2\eta + LZ - NX}{Y} = \frac{R^2\zeta + MX - LY}{Z},$$

which are the equations to the locus of the point ξ, η, ζ, that is, of the centres of least principal moments.

If there is a single resultant, let (ξ, η, ζ) be any point in its direction, and we must have

$$Z\eta - Y\zeta = L,$$
$$X\zeta - Z\xi = M,$$
$$Y\xi - X\eta = N.$$

If we multiply these equations by X, Y, Z, respectively, and add, we obtain the condition for a single resultant, viz.

$$(11)\quad LX + MY + NZ = 0;$$

but if we subtract them, two and two, we find

$$MY - NZ = XY\zeta + XZ\eta - 2YZ\xi,$$

or $$3YZ\xi + MY - NZ = YZ\xi + XZ\eta + XY\zeta:$$

similarly, $$3XZ\eta + NZ - LX = YZ\xi + XZ\eta + XY\zeta,$$

$$3XY\zeta + LX - MY = YZ\xi + XZ\eta + XY\zeta.$$

Hence, equating the first three members of these equations, and dividing by XYZ,

$$(12)\quad \frac{\xi - \frac{1}{3}\left(\frac{N}{Y} - \frac{M}{Z}\right)}{X} = \frac{\eta - \frac{1}{3}\left(\frac{L}{Z} - \frac{N}{X}\right)}{Y} = \frac{\zeta - \frac{1}{3}\left(\frac{M}{X} - \frac{L}{Y}\right)}{Z},$$

which are the equations to the resultant. If we write them for shortness,

$$(13)\quad \frac{\xi - \alpha}{X} = \frac{\eta - \beta}{Y} = \frac{\zeta - \gamma}{Z},$$

[226] it is easily seen that the length of the perpendicular upon it from the origin is

$$\sqrt{\{(\beta Z - \gamma Y)^2 + (\gamma X - \alpha Z)^2 + (\alpha Y - \beta X)^2\}} \cdot \frac{1}{\sqrt{(X^2 + Y^2 + Z^2)}}.$$

If we substitute for α, β, γ in this expression, it may be immediately reduced by the help of equation (11) to the following, viz.

$$\frac{\sqrt{(L^2 + M^2 + N^2)}}{\sqrt{(X^2 + Y^2 + Z^2)}},$$

as we might have anticipated *à priori*.

M. N. N.

Jan. 2, 1839.

NOTE.—In a former Paper (see p. 189, equation 11, &c.) it was stated, that the envelope of the surface under consideration consisted, in certain cases, of three distinct surfaces. But it will be evident on a little reflection, that it really consists only of the points which are common to all the three, as in the case of the ellipsoids there mentioned. I may also observe, that in equation (1) of the same paper, h may be any function of x, y, z not containing a, b, c.

ON THE IMPOSSIBLE LOGARITHMS OF QUANTITIES.

In a paper printed in the fourteenth volume of the *Transactions of the Royal Society of Edinburgh,* I gave a short sketch of what I conceive to be the true nature of Algebra, considered in its greatest generality; that it is the science of symbols, defined not by their nature, but by the laws of combination to which they are subject. In that paper I limited myself to a statement of the general view, without pretending to follow out all the conclusions to which such views would lead us: such an undertaking would be too extended for the limits of a memoir, and would involve a complete treatise on Algebra. It will not, however, be attempting too much to trace out, in one or two cases, some of the more important elucidations which this theory affords of several disputed and obscure points in Algebra, and therefore in the following pages I shall endeavour to point out the deductions which may be derived from the definition of the operation +, given in the paper above alluded to. I there stated, that we must not consider it merely as an affection of other symbols which we call symbols of quantity, but as a distinct operation possessing certain properties peculiar to itself, and subject, like the more ordinary symbols, to be acted on by any other operations, such as the raising to powers, &c. The definition of the operation represented by this symbol is, that [227]

$$+\,+ = +,$$

which leads to the equation

$$(+)^r = +,$$

r being any integer. And this peculiarity—that the operation repeated any number of times gives the same result as when only performed once—is the origin of certain analytical anomalies, which do not at first sight appear to be connected.

The first of these is the fact of the existence of a plurality of roots of a quantity, when the corresponding powers have only one value. It seems a fair question, to ask the cause of so great a difference between two operations so analogous in their nature, but it is one which I have not seen anywhere discussed. The distinction is, I conceive, to be traced to the nature of the operation +, according to the definition of it which I have given; and much of the obscurity connected with the subject is due to an oversight, by which the existence of this + is wholy overlooked. For it is not a, but $+ a$, which has a plurality of roots: and though these quantities are usually reckoned to be the same, this idea is founded on an

illegitimate extension of a supposed relation in the science of number. I say *supposed*, because I hold, that even in Arithmetic a and $+a$ are different, and ought not to be confounded —the former being an absolute operation, the other always a relative one, and consequently incapable of existing by itself. But however this may be, there is no doubt that it is entirely illegitimate to suppose that in all cases a and $+a$ are the same, since generally we know not even what their meanings may be. Indeed, in Geometry the distinction is pretty broadly marked, since a represents a line considered with reference to magnitude only, $+a$ with reference both to magnitude and direction. I therefore maintain, that in general symbolical Algebra we must never consider these quantities as identical; and if at any time we conceive the existence of the $+$, we must take cognizance of its existence throughout all our processes, subjecting it to the operations we may perform on the compound quantity. Now, that in the usual theory of the plurality of roots the existence of $+$ is supposed, though not always expressed, is easily shown from the very first case of plurality of values which occurs. It is argued that, since $a \times a = a^2$ and $-a \times -a = +a^2$ also, we have two values, a and $-a$, for $(a^2)^{\frac{1}{2}}$. But this, it will be seen, depends on the supposition that $+a^2 = a^2$, since in the case of the product $-a \times -a$ the $+$ is exhibited. If, instead of saying $a \times a = a^2$, we were to say that $+a \times +a = +a^2$, we should have undoubtedly $+a^2$ as the result in both cases, and we are therefore entitled to say that $(+a^2)^{\frac{1}{2}}$ has two values, $+a$ and $-a$. The reason for this plurality is now very plain, for

$$(+a^2)^{\frac{1}{2}} = +^{\frac{1}{2}}(a^2)^{\frac{1}{2}} = +^{\frac{1}{2}}a.$$

[228] But from the definition of $+$ it appears that $+^{\frac{1}{2}}$ will be different according as we suppose the $+$ to be equivalent to the operation repeated an even or an odd number of times. In the former case it will be equal to $+$, in the latter to $-$. And generally, if we raise $+a$ to any power m, whether whole or fractional, we have

$$(+a)^m = +^m a^m.$$

Now, as from the definition of $+$ it appears that $+^r = +$, r being any integer, it is indeterminate which power of $+$ it may represent in any case, and therefore we must substitute $+^r$ for $+$, and then, assigning all integer values to r, discover how many values $+^{rm}a^m$ will acquire. So long as m is an integer, rm is an integer, and $+^{rm}a^m$ has only one value; but if m be a fraction of the form $\frac{p}{q}$, $+^{r\frac{p}{q}}$ will acquire different values, according

to the different values we assign to r. It will not, however, acquire an infinite number of values, since after r receives the value q, the values will recur in the same order. Hence the number of values of a quantity raised to a fractional power, is equal to the number of digits in the denominator of the index. It is to be observed, that we must never make $r = 0$, since that assumption is equivalent to supposing that the operation + is not performed at all, which is contrary to our original supposition. From this we see, that the reason why there is a plurality of values for the roots of a quantity, is to be found in the nature of the operation +; and that it is only the compound operation $+ a$, which admits of this plurality, a itself having only one value for each root. This view serves to explain an apparent difficulty which is noticed by various writers on Algebra. Since by the rule of signs $- \times -$ gives +, we ought to have

$$\sqrt{(-a)} \times \sqrt{(-a)} = \sqrt{(+a^2)} = \pm a;$$

whereas we know that it must be only $-a$.

Now this fallacy arises from the sign of the root not being made to affect the + as well as the a. The process is really this,

$$\sqrt{(-a)} \times \sqrt{(-a)} = \sqrt{(+a^2)} = \sqrt{(+)}\,\sqrt{(a^2)} = -a;$$

for in this case we know how the + has been derived, namely from the product $- - = +$ or $-^2 = +$, which of course gives us $+^{\frac{1}{2}} = -$, there being here nothing indeterminate about the +.

It was in consequence of sometimes tacitly assuming the existence of +, and at another time neglecting it, that the errors in various trigonometrical expressions arose; and it was by the introduction of the factor $\cos 2r\pi + -^{\frac{1}{2}} \sin 2r\pi$ (which is equivalent to $+^r$) that Poinsot established the formulæ in a more correct and general shape. Thus the theorem of Demoivre that

$$(\cos \theta + -^{\frac{1}{2}} \sin \theta)^m = \cos m\theta + -^{\frac{1}{2}} \sin m\theta$$

should be written [229]

$$\begin{aligned}\{+^r(\cos \theta + -^{\frac{1}{2}} \sin \theta)\}^m \{&= +^{rm} (\cos \theta + -^{\frac{1}{2}} \sin \theta)^m \\ &= (\cos 2r\pi + -^{\frac{1}{2}} \sin 2r\pi)^m (\cos \theta + -^{\frac{1}{2}} \sin \theta)^m \\ &= \{\cos (2r\pi + \theta) + -^{\frac{1}{2}} \sin (2r\pi + \theta)\}^m \\ &= \cos m\,(2r\pi + \theta) + -^{\frac{1}{2}} \sin m\,(2r\pi + \theta).\end{aligned}$$

It will be seen from what I have said that I suppose the symbol + to play the same part which Professor Peacock ascribes to the symbol 1, when he says that it is the recipient of the affections of a^m; and that what that author considers to be the roots of unity I conceive to be the roots of +.

So far as the correctness of the formulæ is concerned, it makes but little difference which view is taken, if attention be always paid to the existence of this quantity on which the plurality of values depends, whether we denote it by the symbol 1 or +. But in the general Theory of Algebra there is a considerable difference; for 1 being an arithmetical symbol necessarily recalls arithmetical notions; and as the circumstances in which its peculiar nature is evolved occur in general symbolical Algebra, and may be wholly independent of arithmetic, it is of importance to avoid the confusion which must be caused by the introduction into general symbolical Algebra of symbols limited in their signification.

The other point which I propose to elucidate at present, and which is the chief object of this paper, is the plurality of logarithms of quantities, which, although, at first sight unconnected with what we have been discussing, will be found to depend also on the existence of a +, which is generally overlooked. This is closely connected also with the discussion concerning the logarithms of negative quantities, which attracted so much attention in the time of Euler, D'Alembert, and John Bernouilli, and the interest of which has been revived of late years by the researches of Vincent, Ohm, and Graves. Euler had apparently set the question at rest by demonstrating the existence of an infinite number of logarithms of a quantity, one only of which is possible; and the formula he gave was that

$$\log a = L(a) + 2r\pi \sqrt{(-1)},$$

representing by $L(a)$ the arithmetical logarithm of a.

Mr. Graves, by a different and very circuitous process, arrives at the result

$$\log (a) = \frac{L(a) + 2r\pi \sqrt{(-1)}}{1 + 2r'\pi \sqrt{(-1)}},$$

the logarithms being taken with respect to the base ε for simplicity.

The correctness of this result is doubted by Professors Peacock and De Morgan, but it is corroborated by the researches of Sir W. Hamilton and Mr. Warren, as well as of M. Ohm. It is therefore both an interesting and an important question to determine which is the correct result, or at least to point
[230] out the cause of the differences between them. This I think the system I am advocating is able to do. But it is necessary first to lay down distinctly what is the meaning of the operation denoted by *log*; and this, according to my

system, is done by defining its laws of combination. These are

$$\log x + \log y = \log (xy) \dots\dots\dots\dots (1),$$

$$\log (x^y) = y \log x \dots\dots\dots\dots (2),$$

where x and y are distributive and commutative operations,

$$\log a = 1 \dots\dots\dots\dots\dots\dots (3),$$

which assumes the species to be that in which the base is a.

The first and third of these laws are the same as those given by Mr. Graves at the suggestion of Sir William Hamilton, but the second he has omitted; I know not whether from oversight, or from considering it to be unnecessary. I have retained it as I conceive it essential for a strict definition of the operation.

This being premised, I proceed to state the position which I lay down, and the truth of which I hope to be able to establish. It is, that the impossible parts of the general logarithms, whether of those given by Euler or by Mr. Graves, are the logarithms of the symbol + which generally is overlooked in the expressions we use; and that the cause of the difference between the two formulæ for logarithms is, that in that of Euler *one* latent + only, and in that of Mr Graves *two* are exhibited.

This I think is almost apparent from Euler's own process, if we attend to the meaning of the symbols he employs. He substitutes for the number y the expression

$$\{\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi\}\, y,$$

r being any integer which he considers to be equivalent to it; and then taking the logarithms with respect to ε, he says that

$$\log y = L\,(y) + \log \{\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi\},$$

where $L\,(y)$ represents the arithmetical logarithm of y: and as

$$\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi = \varepsilon^{2r\pi\sqrt{(-1)}},$$

we have $$\log y = L\,(y) + 2r\pi \sqrt{(-1)}.$$

As r may receive any integer value, this expression has an infinite number of values, one only of which is possible in the case when $r = 0$. It will be seen that the correctness of this result depends essentially on the assumption that y and $\{\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi\}\, y$ are identical: an assumption which at first it seems very natural to make, since the expression $\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi$ is usually considered to be equal to unity. But if we suppose the quantities with which we are dealing to be general quantities, and not numbers merely, a numerical value of $\cos 2r\pi + \sqrt{(-1)} \sin 2r\pi$ can have no place in our investigation, and we must seek for its general alge-

These last he seems quite to have overlooked, which may have arisen from his having adopted, with many other mathematicians, the name of *imaginary* quantities. I adhere to the name *impossible* instead of *imaginary*, because the latter involves an idea which I conceive to be very deleterious in analysis. We may be unable to perform an operation though it be by no means an *imaginary* one; and indeed all that we can say of those quantities which have this name affixed to them is, that they are *uninterpretable in arithmetic*. For this reason, if I were permitted to propose a change, I should prefer to call these quantities "operations uninterpretable in arithmetic;" as this involves no theory of their nature, but only expresses what is a fact.

[233] That, according to the system which I adopt, there cannot be a logarithm common to both positive and negative quantities, is easily shown. A positive quantity may be generally expressed by $+^{r}a$:

the logarithm of which is

$$\log a + \log (+^{r}) = \log a + 2r\pi \sqrt{(-1)}.$$

A negative quantity may be expressed by

$$+^{\frac{2r+1}{2}} a;$$

the logarithm of which is

$$\log a + \log (+^{\frac{2r+1}{2}}) = \log a + \frac{2r+1}{2} \pi \sqrt{(-1)}.$$

And these two expressions can never coincide; nor can either ever lose its impossible part, since we are not at liberty to make $r = 0$ in the first case, or $= -\frac{1}{2}$ in the second.

It is somewhat remarkable, that Mr. Peacock has been led into the same error as M. Vincent and Mr. Graves, respecting the coincidence in some cases of the logarithms of positive and negative quantities. As the cause of his error has reference to the remark which I have just made, and is not very easy to be detected, I shall point it out more particularly.

He considers $-a^{m}$ to be equivalent to $-1\,(+a)^{m}$, which gives

$$\log -(a)^{m} = \log(-1) + \log(+a)^{m}$$
$$= (2r + 2mr' + 1)\,\pi \sqrt{(-1)} + m \log a.$$

He then supposes $m = \dfrac{p}{2n}$ where p is prime to n, $r' = -n$, and $r = \dfrac{p-1}{2}$; and as these values make the multiplier of $\pi \sqrt{(-1)}$ vanish, he concludes that the logarithm of $-(a)^{m}$ coincides

with that of a^m, since it becomes $m \log a$. Now on this it is to be observed, that since m affects the + in $(+a)^m$, $-a^m$ is really equal to $-1 \,.\, +^m \,.\, a^m$, or, putting the general values for $-$ and +, to

$$+^{\frac{2r+1}{2}} (+^{r'})^m a^m.$$

In this expression, if we make $m = \dfrac{p}{2n}$, $r' = -n$, $r = \dfrac{p-1}{2}$, it becomes

$$+^{\frac{p}{2}} \,.\, +^{-\frac{p}{2}} \,.\, a^{\frac{p}{2n}};$$

and as $+^{\frac{p}{2}}$ and $+^{-\frac{p}{2}}$ are inverse operations, they destroy each other, and we have simply $a^{\frac{p}{2n}}$; the logarithm of which is, as it should be, possible. But these assumptions as to the values of m, r, and r', are plainly not allowable, since they imply, as we have seen, that a^m is not affected by $-$ at all, which [234] is contrary to the original supposition. Hence we perceive that Mr. Peacock's argument for the existence of logarithms common to positive and negative quantities, being based on an unlawful assumption, falls to the ground.

If it be allowable to assume any quantity as base for a system of logarithms, we might, instead of $+^r a$ when r is an integer, take the same quantity, supposing r to be a fraction. We should then have possible quantities corresponding to impossible logarithms, and impossible quantities to possible logarithms; but the subject does not appear to be of sufficient interest to require an extended discussion.

In conclusion, I will recapitulate the conclusions to which I have been led by this mode of considering the symbol +.

1. A simple distributive and commutative operation has only one root, but if it be compounded with + it has a plurality of roots depending on the indeterminate nature of +.

2. If the base of a system of logarithms and the number be simple distributive and commutative operations, there is only one corresponding logarithm; but if the number of the form be $+^r y$, there is an infinite number of logarithms.

3. If the base of the system be of the form $+^r a$, we are only allowed to assign one value to r, (as otherwise we alter the system,) and then there will be no plurality of logarithms.

4. The logarithms of $+a$ and $-a$ are in all cases different, and neither ever coincide with that of a.

5. The impossible parts of the logarithms, as usually given, are the logarithms of + and of $-$.

Trinity College.

ON THE EQUATION WHICH DETERMINES THE STABILITY OF THE PLANETARY EXCENTRICITIES.

THE equation which enables us to prove the stability of the excentricities of the planetary orbits, may be deduced as follows.

We have

$$\frac{de_1}{dt} = -\frac{n_1 a_1}{\mu e_1}(1 - e_1^2)\frac{dR_1}{d\varepsilon_1} + \frac{n_1 a_1}{\mu e_1}\sqrt{(1 - e_1^2)}\left(\frac{dR_1}{d\varepsilon_1} + \frac{dR_1}{d\omega_1}\right)$$

$$= \frac{n_1 a_1}{\mu e_1}(1 - e_1^2)\frac{dR_1}{d\varepsilon_1} + \frac{n_1 a_1}{\mu e_1}\sqrt{(1 - e_1^2)}\frac{dR_1}{d\theta_1}.$$

[235] Now, wherever ε_1 occurs in R_1, $n_1 t + \varepsilon_1$ occurs, that is, ε_1 occurs only in periodical terms; hence, neglecting periodical terms,

$$\frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} = \frac{m}{\mu}\sqrt{(1 - e_1^2)}\frac{dR_1}{d\theta_1};$$

or, neglecting terms of a higher order than the second, and observing that $\frac{m}{n_1 a_1} e_1 \frac{de_1}{dt}$ is itself of the first order, we have

$$\frac{m}{n_1 a} e_1 \frac{de_1}{dt} = \frac{m}{\mu}\frac{dR}{d\theta}.$$

Now,

$$R = -\frac{m'}{\sqrt{\{r^2 - 2rr'\cos(\theta - \theta') + r'^2\}}} + \frac{m'r\cos(\theta - \theta')}{r'^2};$$

$$\therefore \frac{dR}{d\theta} = -\frac{m'\sin(\theta - \theta')rr'}{\{r^2 - 2rr'\cos(\theta - \theta') + r'^2\}^{\frac{3}{2}}} - \frac{m'r\sin(\theta - \theta')}{r'^2}:$$

$$\therefore \frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} = k\sin(\theta - \theta') - \frac{mm'}{\mu}\frac{r\sin(\theta - \theta')}{r'^2},$$

where k is a quantity the same for both m and m'.

Similarly,

$$\frac{m'}{n_1' a_1'} e_1' \frac{de_1'}{dt} = -k\sin(\theta - \theta') + \frac{mm'}{\mu}\frac{r'\sin(\theta - \theta')}{r^2};$$

$$\therefore \frac{m}{n_1 a_1} e_1 \frac{de_1}{dt} + \frac{m'}{n_1' a_1'} e_1' \frac{de_1'}{dt} = -\frac{mm'}{\mu}\left(\frac{r}{r'^2} - \frac{r'}{r^2}\right)\sin(\theta - \theta'),$$

we have

$$\sin(\theta - \theta') = \sin\{nt + \varepsilon - (n't + \varepsilon') + z\}$$

$$= \sin\{nt + \varepsilon - (n't + \varepsilon')\}\cos z + \cos\{nt + \varepsilon - (n't + \varepsilon')\}\sin z,$$

where $\begin{matrix}\sin\\ \cos\end{matrix} z$ consist each of a series of terms respectively of the form

$$P \begin{matrix}\sin\\ \cos\end{matrix} \{p(nt + \epsilon - \varpi) - q(n't + \epsilon' - \varpi')\},$$

where P is a quantity involving $e^p e'^q$; and $\frac{r}{r'^2}$, $\frac{r'}{r^3}$ each consists of a series of terms of the form

$$P \cos \{p(nt + \epsilon - \varpi) - q(n't + \epsilon' - \varpi')\},$$

where P involves $e^p e'^q$, so that $\cos z \cdot \frac{r}{r'^2}$ and $\cos z \cdot \frac{r'}{r^3}$ will be made up of terms of the form

$$P \cos \{p(nt + \epsilon - \varpi) - q(n't + \epsilon' - \varpi')\};$$

and $\sin z \cdot \frac{r}{r'^2}$, as also $\sin z \cdot \frac{r'}{r^3}$, will be made up of terms of the form

$$P \sin \{p(nt + \epsilon - \varpi) - q(n't + \epsilon' - \varpi')\}, \qquad [236]$$

P in each case involving $e^p e'^q$. Now the only way in which a constant term can arise in $\sin(\theta - \theta') \frac{r}{r'^2}$, is by the combination of $\begin{matrix}\sin\\ \cos\end{matrix} \{nt + \epsilon - (n't + \epsilon')\}$ with $\begin{matrix}\sin\\ \cos\end{matrix} \{nt + \epsilon - \varpi - (n't + \epsilon' - \varpi')\}$, so that $\sin(\theta - \theta') \frac{r}{r'^2}$ can only contain one such term, which will be of the form

$$ee' \{A + \phi(e, e')\} \cos(\varpi - \varpi')\},$$

where ϕ denotes an integral function. Similarly, $\sin(\theta - \theta') \frac{r}{r^3}$ can only contain one constant term, which will be

$$ee' \{A + \psi(e, e')\} \cos(\varpi - \varpi');$$

hence, rejecting quantities of a higher order than the second, there is no constant term in $\sin(\theta - \theta') \left(\frac{r}{r'^2} - \frac{r}{r^2}\right)$;

$$\therefore \frac{m}{na} e_1 \frac{de_1}{dt} + \frac{m'}{n'a'} e_1' \frac{de_1'}{dt} = 0;$$

$$\therefore \frac{m}{na} e_1^2 + \frac{m'}{n'a'} e_1'^2 = C.$$

λ.

GENERAL FORMULÆ FOR THE CHANGE OF THE INDEPENDENT VARIABLE.

Given an expression involving the differential coefficients of u with respect to x, it is required to change it into an equivalent expression involving the differential coefficients of u with respect to y, y being a given function of x.

Suppose $u = F(y)$; then, a being an arbitrary quantity,

$$F(y) = F\{a + (y - a)\}$$

$$= F(a) + \frac{y - a}{1} F_1(a) + \frac{(y - a)^2}{1.2} F_2(a) + \frac{(y - a)^3}{1.2.3} F_3(a) + \ldots$$

$$\therefore \frac{d^n F(y)}{dx^n} = \frac{1}{1} \frac{d^n (y - a)}{dx^n} F_1(a) + \frac{1}{1.2} \frac{d^n (y - a)^2}{dx^n} F_2(a) + \frac{1}{1.2.3} \frac{d^n (y - a)^3}{dx^n} F_3(a) + \ldots$$

Suppose the differentiation performed, and then put $a = y$;

[237] $$\frac{d^n u}{dx^n} = \frac{1}{1} \frac{du}{dy} \frac{d^n (y - a)}{dx^n} + \frac{1}{1.2} \frac{d^2 u}{dy^2} \cdot \frac{d^n (y - a)^2}{dx^n} + \frac{1}{1.2.3} \frac{d^3 u}{dy^3} \cdot \frac{d^n (y - a)^3}{dx^n} + \ldots$$

where y is to be substituted for a after differentiation. The number of terms will be finite, because the n^{th} differential coefficients of the powers of $y - a$ higher than the n^{th} will vanish when y is put for a.

It remains to express $\frac{d^n (y - a)^p}{dx^n}$ in terms of the differential coefficient of y with regard to x.

Let y' be the value of y when x is changed to $x + h$. Then, by Taylor's theorem,

$$y' - a = y - a + \frac{h}{1} \frac{dy}{dx} + \frac{h^2}{1.2} \frac{d^2 y}{dx^2} + \frac{h^3}{1.2.3} \frac{d^3 y}{dx^3} + \ldots$$

also

$$(y' - a)^p = (y - a)^p + \frac{h}{1} \frac{d(y - a)^p}{dx} + \frac{h^2}{1.2} \frac{d^2 (y - a)^p}{dx^2} + \frac{h^3}{1.2.3} \frac{d^3 (y - a)^p}{dx^3} + \ldots$$

Hence $\frac{d^n (y - a)^p}{dx^n} = 1.2 \ldots n \times$ coefficient of h^n in

$$\left(y - a + \frac{n}{1} \frac{dy}{dx} + \frac{h^2}{1.2} \frac{d^2 y}{dx^2} + \frac{h^3}{1.2.3} \frac{d^3 y}{dx^3} + \ldots\right)^p.$$

Now, by the polynomial theorem, the coefficient of h^n in the expansion of this is the sum of the terms

$$\frac{1.2.3..p}{(1\,2.\,...\,a)\,(1.2..\beta)\,(1.2..\gamma)\,..}$$
$$\times\left(\frac{1}{1.2\,...\,\lambda}\,\frac{d^\lambda y}{dx^\lambda}\right)^a\left(\frac{1}{1.2..\mu}\,\frac{d^\mu y}{dx^\mu}\right)^\beta\left(\frac{1}{1.2..\nu}\,\frac{d^\nu y}{dx^\nu}\right)^\gamma\,...$$

where $\qquad a + \beta + \gamma + \ldots\ldots = p;$

and $\qquad a\lambda + \beta\mu + \gamma\nu + \ldots\ldots = n,$

therefore the general term of $\dfrac{d^n\,(y-a)^p}{dx^n}$ is

$$\frac{1.2.3\ldots.p\times 1.2.3\ldots.n}{1.2...a\times 1.2...\beta\times 1.2...\gamma\times\ldots\times(1.2...\lambda)^a\,(1.2...\mu)^\beta\,(1.2...\nu)^\gamma\,...}$$
$$\times\left(\frac{d^\lambda y}{dx^\lambda}\right)^a\left(\frac{d^\mu y}{dx^\mu}\right)^\beta\left(\frac{d^\nu y}{dx^\nu}\right)\,...$$

the quantities a, β, γ, &c. and λ, μ, ν, &c. being subject to the two conditions above. It must be observed, that for $a = 0$, $1.2\,...\,a$ becomes 1, and for $\lambda = 0$, $1.2\,...\,\lambda$ becomes 1, and $\dfrac{d^\lambda y}{dx^\lambda}$ becomes $y - a$. Hence, the terms in which [238]
any of the quantities λ, μ, ν, &c. are 0, may be neglected, because they vanish when y is put for a.

S. S. G.

MATHEMATICAL NOTES.

1. The following is an easy method of obtaining the general differences of sin x and cos x.

$$\Delta\sin x = \sin(x+h) - \sin x = \sin x\,(\cos h - 1) + \cos x\sin h$$
$$= 2\sin\frac{h}{2}\left(-\sin x\sin\frac{h}{2} + \cos x\cos\frac{h}{2}\right).$$

But $-\sin x = \cos\left(x + \dfrac{\pi}{2}\right)$. $\cos x = \sin\left(x = \dfrac{\pi}{2}\right)$;

therefore

$$\Delta\sin x = 2\sin\frac{h}{2}\left\{\cos\left(x+\frac{\pi}{2}\right)\sin\frac{h}{2} + \sin\left(x+\frac{\pi}{2}\right)\cos\frac{h}{2}\right\}$$
$$= 2\sin\frac{h}{2}\sin\left(x + \frac{\pi+h}{2}\right) = \left(2\sin\frac{h}{2}\right)D^{\frac{1}{2}(\pi+h)}\sin x;$$

therefore
$$\Delta^n \sin x = \left(2\sin\frac{h}{2}\right)^n D^{\frac{n}{2}(\pi+h)} \sin x$$
$$= \left(2\sin\frac{h}{2}\right)^n \sin\left\{x + \frac{n}{2}(\pi + h)\right\}.$$

Similarly, $$\Delta^n \cos x = \left(2\sin\frac{h}{2}\right)^n \cos\left\{x + \frac{n}{2}(\pi + h)\right\}.$$

These results are true whether n be positive or negative, whole or fractional.

In differentials the same method may be employed,
$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \frac{\pi}{2}\right) = D^{\frac{\pi}{2}} \sin x,$$
and therefore
$$\left(\frac{d}{dx}\right)^n \sin x = D^{n\frac{\pi}{2}} \sin x = \sin\left(x + n\frac{\pi}{2}\right),$$
whatever n may be.

β

[239] 2. It seems not to be generally known, that the equation
$$1.2.3.\ldots.n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots.$$
which is used in proving Sir John Wilson's theorem respecting prime numbers, can be deduced immediately from the theorems of common Algebra. The following is the method.

By the Binomial Theorem,
$$(\varepsilon^x - 1)^n = \varepsilon^{nx} - \frac{n}{1}\varepsilon^{(n-1)x} + \frac{n(n-1)}{1.2}\varepsilon^{(n-2)x} - \ldots.$$
Substitute for each exponential its expansion according to powers of x, and equate the coefficients of x^n on the two sides. That on the first side, or $\left(x + \frac{x^2}{1.2} + \ldots.\right)^n$, is evidently 1; the coefficient of x^n in ε^{nx} is $\frac{n^n}{1.2.3.\ldots.n}$; in $\varepsilon^{(n-1)x}$, $\frac{(n-1)^n}{1.2.3.\ldots.n}$, in $\varepsilon^{(n-2)x}$, $\frac{(n-2)^n}{1.2.3.\ldots.n}$, &c. Hence,
$$1 = \frac{n^n}{1.2.3.\ldots.n} - \frac{n}{1}.\frac{(n-1)^n}{1.2.3.\ldots.n} + \frac{n(n-1)}{1.2}\,\frac{(n-2)^n}{1.2.3.\ldots.n} - \ldots.$$
or $$1.2.3.\ldots.n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots.$$

In the same way it is seen that

$$n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \ldots.$$

is zero if m and n be any integer, of which n is the greater.

γ.

3. The equation to the tangent of the ellipse given in Art. 2, of No. I., furnishes a ready solution of the problem. To find the locus of the intersections of pairs of tangents to an ellipse, which are always parallel to conjugate diameters.

The equation to one tangent being

$$y - \alpha x = \sqrt{(a^2\alpha^2 + b^2)};$$

that to the other is

$$y + \frac{b^2}{a^2\alpha} = \sqrt{\left(\frac{a^2b^4}{a^4\alpha^2} + b^2\right)},$$

since it is parallel to the conjugate diameter. Multiplying up, this becomes

$$a^2\alpha y + b^2x = ab\sqrt{(b^2 + a^2\alpha^2)};$$

and multiplying the first equation by ab, it becomes

$$aby - ab\alpha x = ab\sqrt{(a^2\alpha^2 + b^2)}.$$

Squaring these two equations, and adding, we get [240]

$$a^2y^2(a^2\alpha^2 + b^2) + b^2x^2(a^2\alpha^2 + b^2) = 2a^2b^2(a^2\alpha^2 + b^2),$$

or

$$a^2y^2 + b^2x^2 = 2a^2b^2.$$

δ.

4. *Decomposition of Rational Fractions.*—If the denominator of a rational fraction contain equal roots, the equivalent fractions may be easily determined by a process similar to Maclaurin's theorem.

Let the fraction be $\dfrac{f(x)}{(x-a)^n}$; then

$f(x) = f(x + z - a)$ (when $z = a$) $= f(z + x - a)$ (when $z = a$), or, expanding by Taylor's theorem,

$$f(x) = f(z) + (x-a)\frac{d}{dz}f(z) + \ldots \frac{(x-a)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}f(z) + R,$$

(when $z = a$).

If $f(x)$ be an integral function of x of a degree, at least one less than the degree of the denominator, $R = 0$, since all the terms after the n^{th} vanish. If $f(x)$ be fractional, we must

determine R in the usual way. Dividing now by $(x-a)^n$, we have

$$\frac{f(x)}{(x-a)^n} = \frac{f(z)}{(x-a)^n} + \frac{1}{(x-a)^{n-1}}\frac{d}{dz}f(z) + \dots$$
$$+ \frac{1}{(n-1)!\,(x-a)}\frac{d^{n-1}}{dz^{n-1}}f(z) + \frac{R}{(x-a)^n},$$
$$(\text{when } z = a).$$

ϕ.

5. Note on Art. VII.—The view taken in this article of the cause of the plurality of values of a root of $+a$, may perhaps be more clearly explained by stating, that the idea entertained is, that there is really no plurality of *roots* of one quantity, but that there is an indeterminateness as to the quantity, the root of which is taken: and the same is to be said of the logarithms.

ϕ.

[241] ON THE VIBRATIONS OF A MUSICAL STRING.

THE tension of the string being supposed very great compared with its weight, in the position of equilibrium, the string will be as nearly as possible rectilinear; we shall here suppose it accurately so.

Let ξ be the distance of any point of the string from one extremity when the string is at rest, and let x, y, z be the three coordinates of the same point at the time t, x having the same origin and direction with ξ, and y and z being perpendicular to x and to each other.

It is plain that x, y, z are functions of ξ and t; hence a point, which when the string is at rest is situated at a distance $\xi + \delta\xi$ from the origin, will, at the time t, have its three coordinates respectively equal to

$$x + \frac{dx}{d\xi}\delta\xi, \quad y + \frac{dy}{d\xi}\delta\xi, \quad z + \frac{dz}{d\xi}\delta\xi;$$

$$\frac{dx}{d\xi}, \ \frac{dy}{d\xi}, \text{ and } \frac{dz}{d\xi},$$

representing the partial differential coefficients with respect to ξ, t being regarded as constant.

Hence, if δs be the distance of the two points of the string abovementioned at the time t, and α, β, γ the angles which δs makes with the three axes, we shall have

$$\delta s = \delta\xi \sqrt{\left(\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}\right)}, \qquad [242]$$

$$\cos\alpha = \frac{\frac{dx}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dx}{d\xi}}{\sqrt{\left(\frac{dx^2}{dy^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}\right)}},$$

$$\cos\beta = \frac{\frac{dy}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dy}{d\xi}}{\sqrt{\left(\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}\right)}},$$

$$\cos\gamma = \frac{\frac{dz}{d\xi}\,\delta\xi}{\delta s} = \frac{\frac{dz}{d\xi}}{\sqrt{\left(\frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} + \frac{dz^2}{d\xi^2}\right)}}.$$

The above expressions are general whatever be the extent of the vibrations. But if we limit our investigation to the case where the vibrations are extremely small, putting $x = \xi + u$, the quantities u, y, z, as well as their differential coefficients, are all extremely small. Hence, writing for $\frac{dx}{d\xi}$ its value $1 + \frac{du}{d\xi}$, and neglecting such magnitudes as may legitimately be neglected consistently with this hypothesis, we shall have very approximately,

$$\delta s = \delta\xi\left(1 + \frac{du}{d\xi}\right),$$

$$\cos\alpha = 1, \quad \cos\beta = \frac{dy}{d\xi}, \quad \cos\gamma = \frac{dz}{d\xi}.$$

Hence, if T_1 be the tension of the string at the time t at the former of the two points of the string under consideration, or at the extremity of δs nearest the origin of the coordinates, the three components $T_1 \cos\alpha$, $T_1 \cos\beta$, $T_1 \cos\gamma$, will be very nearly expressed by

$$T_1, \quad T_1\frac{dy}{d\xi}, \text{ and } T_1\frac{dz}{d\xi}.$$

At the other extremity of δs these three components will be respectively increased by

$$\frac{dT_1}{d\xi}\cdot\delta\xi,\quad \frac{d}{d\xi}\left(T_1\frac{dy}{d\xi}\right)\cdot\delta\xi,\quad\text{and}\quad \frac{d}{d\xi}\left(T_1\frac{dz}{d\xi}\right)\cdot\delta\xi;$$

which three expressions are therefore the moving forces which tend to move the particle δs parallel to the three axes, and to *increase* the coordinates x, y, z.

[243] Let W be the weight of a *unit of length* of the string in the state of rest. Then $\frac{W}{g}$ is the mass of that portion of string: consequently $\frac{W\delta\xi}{g}$ is the mass of the portion $\delta\xi$, which is the same as that of δs at the time t. Hence, instituting the equation of motion, and writing for $\frac{d^2x}{dt^2}$ its equivalent $\frac{d^2u}{dt^2}$, we get

$$\frac{d^2u}{dt^2}=\frac{g}{W}\cdot\frac{dT_1}{d\xi},$$

$$\frac{d^2y}{dt^2}=\frac{g}{W}\cdot\frac{d}{d\xi}\left(T_1\frac{dy}{d\xi}\right),$$

$$\frac{d^2z}{dt^2}=\frac{g}{W}\cdot\frac{d}{d\xi}\left(T_1\frac{dz}{d\xi}\right).$$

Let T be the tension of the whole string when at rest; then T_1 is manifestly a function of the ratio $\frac{\delta s}{\delta\xi}$, which reduces itself to T when $\delta s=\delta\xi$, or when $\frac{\delta s}{\delta\xi}-1=0$; that is, when $\frac{du}{\delta\xi}=0$, $\left\{\text{since } \delta s=\delta\xi\cdot\left(1+\frac{du}{d\xi}\right)\right\}$. If, therefore, we regard T_1 expanded in powers of $\frac{\delta s}{d\xi}-1$ or $\frac{du}{\delta\xi}$, we shall have $T_1=T+Q\,\frac{du}{d\xi}$, neglecting the higher powers of $\frac{du}{d\xi}$. The constant Q must be determined experimentally.

Now it appears by experiment, that within certain limits the increase of tension is proportional to the increase of length. Suppose the unit of length, whose tension is originally T, to be pulled out to the length $1+\lambda$, and let P, its increase of tension, be observed; the portion $\delta\xi$ would be extended to the

length $\delta\xi \,.\, \frac{1 + \lambda}{1}$; but when $\delta\xi$ is extended to the length δs, the increase of tension is $T_1 - T$; therefore

$$\frac{T_1 - T}{P} = \frac{\delta s - \delta\xi}{\delta\xi\,(1 + \lambda) - \delta\xi} = \frac{1}{\lambda} \,.\, \left(\frac{\delta s}{\delta\xi} - 1\right);$$

$$\therefore\; T_1 = T + \frac{P}{\lambda}\left(\frac{\delta s}{\delta\xi} - 1\right)$$

$$= T + \frac{P}{\lambda} \,.\, \frac{du}{d\xi};$$

hence $Q = \frac{P}{\lambda}$, and is therefore known.

[244]

If in the three equations of motion we write for T_1 its value $T + Q \,.\, \frac{du}{d\xi}$, and neglect the products $\frac{dy}{d\xi} \,.\, \frac{du}{d\xi}$ and $\frac{dz}{d\xi} \,.\, \frac{du}{d\xi}$, they will become

$$\frac{d^2u}{dt^2} = b^2 \,.\, \frac{d^2u}{d\xi^2},$$

$$\frac{d^2y}{dt^2} = a^2 \,.\, \frac{d^2y}{d\xi^2},$$

$$\frac{d^2z}{dt^2} = a^2 \,.\, \frac{d^2z}{d\xi^2},$$

where $b^2 = \frac{Q}{W}\, g$, and $a^2 = \frac{T}{W}\, g$.

The form of these three equations of motion shows, that the vibrations parallel to the three axes take place independently of each other.

Let us first consider those parallel to the axes of y, which are given by the equation

$$\frac{d^2y}{dt^2} = a^2\, \frac{d^2y}{dx^2},$$

in which ξ has been changed into x: no confusion will arise from this change, as we shall not again use the latter symbol in its original signification. The integral of the last equation is

$$y = f(x + at) + F(x - at).$$

It is now well established that the arbitrary functions introduced by the integration of partial differential equations, are not subject to the law of algebraical continuity, but may be regarded as defined by curves drawn *liberâ manu,* subject

however to the condition that the tangents at any two consecutive points shall not make a finite angle with each other.

Let us suppose that the string is initially displaced from its position of equilibrium, for a moment kept at rest in that displaced position, and then abandoned to itself; and let t be measured from the moment of abandonment. The velocity parallel to y being determined by the equation

$$\frac{dy}{dt} = af'(x + at) - aF'(x - at),$$

since, when $t = 0$, every point is at rest, we have $f'(x) = F'(x)$, and consequently $F(x) = c + f(x)$, for all values of x between the extreme points of the string.

Therefore the equation

$$y = f(x + at) + c + f(x - at)$$

will hold for all values of the arguments $x + at$, $x - at$, which fall between o and l, l being the length of the string.

With regard to those values of $F(x - at)$ for which the argument falls *beyond* the limits of the string; since $x - at$ is essentially negative for such values, and consequently distinct from the argument of $f(x + at)$ which is essentially positive for all values, it is clear that $F(x - at)$ may for such values be replaced by the *equally arbitrary* form $c + f(x - at)$. This depends on the arbitrary and discontinuous nature of the form $f(v)$.

It follows from the above that the equation

$$y = \frac{c}{2} + f(x + at) + \frac{c}{2} + f(x - at)$$

or, (writing the form $\frac{1}{2}\phi(v)$ in the place of $\frac{c}{2} + f(v)$), the equation

$$y = \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\phi(x - at)$$

is, without any restriction, the proper solution in the case proposed.

The form of ϕ is known when the initial shape of the string is given; for, making $t = 0$, we have

$$y = \phi(x),$$

and the values of y being given from $x = 0$ to $x = l$, the values of $\phi(v)$ are known from $v = 0$ to $v = l$.

[245] The condition, that the two extremities are fixed, will enable us to determine the value of $\phi(v)$ from $v = -\infty$ to $v = +\infty$. For since the first extremity is fixed for which $x = 0$, we must have

$$0 = \tfrac{1}{2}\phi(at) + \tfrac{1}{2}\phi(-at)$$

for all positive values of t, and consequently

$$\phi(-v) = -\phi(v)$$

for all values of v. The condition that the other extremity is fixed gives us in like manner

$$0 = \tfrac{1}{2}\phi(l + at) + \tfrac{1}{2}\phi(l - at)$$

for all positive values of t, and consequently

$$\phi(l + v) = -\phi(l - v)$$

for all positive values of v. Writing in the last equation $l + v$ in the place of v, we get

$$\phi(2l + v) = -\phi(-v),$$

and therefore $\phi(2l + v) = \phi(v)$.

Hence, the value of $\phi(v)$ from $v = 0$ to $v = l$ being given by the initial shape of the curve, its value from $v = l$ to $v = 2l$ will be given by the equation

$$\phi(l + v) = -\phi(l - v):$$

again, from $v = 2l$ to $v = 4l$, $\phi(v)$ will be given by the equation

$$\phi(2l + v) = \phi(v);$$

and in the same manner, by successive substitutions in the last equation, the value of $\phi(v)$ may be determined for any positive value of v. But $\phi(v)$ being known for all positive values of v, will also be known for all negative values, by means of the equation

$$\phi(-v) = -\phi(v).$$

Hence $\phi(v)$ is known from $v = -\infty$ to $v = +\infty$.

Having thus determined the values of the function $\frac{1}{2}\phi(v)$ for all values of v, positive or negative, the values of $\frac{1}{2}\phi(x + at)$ and $\frac{1}{2}\phi(x - at)$ will be known for all values of x and t; and thus the equation

$$y = \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\phi(x - at)$$

determines the position of every point of the string at any assigned time.

It may be observed, that if we construct the curve [246]
$y = \frac{1}{2}\phi(x)$ from $x = -\infty$ to $x = \infty$, the equation

$$\phi(l + v) = -\phi(l - v)$$

shows, that from $x = l$ to $x = 2l$ the curve is the same as from $x = 0$ to $x = l$, except that it is inverted, both with respect to right and left, and with respect to up and down. Again, from $x = 2l$ to $x = 3l$ it is the same as from $x = 0$ to $x = l$, as appears from the equation

$$\phi(2l + v) = \phi(v).$$

From $x = 3l$ to $x = 4l$, since

$$\phi(3l + v) = \phi(l + v) = -\phi(l - v),$$

the curve will be inverted, as between $x = l$ and $x = 2l$; and the same succession of direct and inverted curve will recur *ad inf.*

On the negative side of the origin, the equation

$$\phi(-v) = -\phi(v)$$

shows, that the same alternation of curves in the inverted and direct position will succeed each other *ad inf.* as on the positive side.

The quantities $x + at$ and $x - at$, represent for a given value of x two lengths of line, the one increasing and the other decreasing with the uniform velocity a. And since $\frac{1}{2}\phi(x + at)$ remains of the same value when x becomes $x - h$ and t becomes $t + \frac{h}{a}$, we see that $\frac{1}{2}\phi(x + at)$ represents an ordinate, which is transferred from the abscissa x to the abscissa $x - h$ in a time $\frac{h}{a}$, and consequently with a velocity $h \div \frac{h}{a}$ or (a). In like manner, since $\frac{1}{2}\phi(x - at)$ remains of the same value when x becomes $x + h$ and t becomes $t + \frac{h}{a}$, $\frac{1}{2}\phi(x - at)$ represents an ordinate, which is transferred from the abscissa x to the abscissa $x + h$ in the time $\frac{h}{a}$, and therefore with the velocity (a).

Since what has just been said is true of all values of x for the same values of the time, namely t and $t + \frac{h}{a}$, it appears that $\frac{1}{2}\phi(x + at)$ represents a form of curve or undulation which is transferred towards the *negative* infinity with the uniform velocity (a); and $\frac{1}{2}\phi(x - at)$ represents a curve or undulation of exactly the same shape as the former, transferred towards the *positive* infinity with the same uniform velocity. The curve which results from the superposition of these two undulations at any instant, is the momentary shape of the vibrating string.

[247] When $t = \frac{2l}{a}$, $y = \frac{1}{2}\phi(x + 2l) + \frac{1}{2}\phi(x - 2l)$,

but $\qquad \phi(x + 2l) = \phi(x),$

and $\phi(x-2l) = -\phi(2l-x) = -\phi\{l+(l-x)\}$

$$= \phi\{l-(l-x)\}, \ \{\text{since } \phi(l+v) = -\phi(l-v)\}$$
$$= \phi(x);$$
$$\therefore y = \phi(x).$$

Hence the string returns to its original shape after a time $\frac{2l}{a}$, which is therefore the time of a complete vibration.

The best method of exhibiting to the eye the changes of shape which the string passes through in the course of each vibration, is actually to draw the curves one below the other, which must be superposed in order to produce the instantaneous form of the string.

Figure (1) represents one vibration divided into four distinct phases of equal duration. The upper of the three lines in each phase represents the string itself vibrating between its extreme points A, B, which are fixed. The second and third lines represent the two constituent forms $\frac{1}{2}\phi(x+at)$ and $\frac{1}{2}\phi(x-at)$.

In the first phase, where $t = 0$, each of the constituent forms has for its equation $y = \frac{1}{2}\phi(x)$, and will be drawn by taking the ordinates half the size of those of the initial figure represented in the upper line, whose equation is $y = \phi(x)$.

In the second line, from $x = l$ to $x = 2l$, the same constituent form is repeated in an inverted position; and from $x = 2l$ to $x = 3l$, the same form is again repeated in the original position.

In the third line the same constituent form is drawn, in its direct position under the vibrating string, then in its inverted position from $x = 0$ to $x = -l$, afterwards in its erect position from $x = -l$ to $x = -2l$.

This construction being made, the position of any point P of the line is easily determined at any time t as follows.

Draw the corresponding ordinates PM, P_1M_1, P_2M_2, in the original curve APB, and its two constituents $A_1P_1B_1$, $B_2P_2A_2$. Take M_1N_1 and M_2N_2, the first on the positive side of M_1 and the second on the negative side of M_2, making them each equal to (at), and draw the ordinates N_1Q_1, N_2Q_2. Then, taking

$$P'M = N_1Q_1 + N_2Q_2,$$

P' will be the place of P at the time t.

It may be observed that N_1, according to the magnitude of t, may fall anywhere in M_1B_1, or M_1B_1 produced; and N_2 anywhere in M_2B_2, or M_2B_2 produced; and when the ordinates N_1Q_1, N_2Q_2 fall on the other side of the axis, they must be reckoned as negative in taking their algebraical sum, and

$P'M$ must be measured above or below the axis, as this algebraical sum is positive or negative.

[248] But instead of setting off the ordinates N_1Q_1, N_2Q_2 along the *fixed* curves $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, we may regard these curves as carried upon two separate sliders, and moved with the uniform velocity (a), the former towards the negative, and the latter towards the positive infinity. In this manner N_1Q_1 and N_2Q_2 would arrive under the point M of the string in the same instant; and to find the form of the whole string at that instant, we shall only have to take all the ordinates between A and B equal to the sum of the ordinates which fall immediately below them, paying proper attention to the signs as before directed.

We have thus an easy and simple method of exhibiting the string in any phase of vibration, and we shall presently see how it puts the subject of the harmonic sounds, and the reflexion of the undulations at the extremities of the string, under the clearest point of view.

In phase II. of fig. (1), each slider has been advanced through a space $\frac{1}{2}l$, and the figure of the string constructed by the rule laid down; and it may be observed in this, as in all the subsequent figures, how the condition of the fixity of the points A and B is satisfied by two equal and opposite ordinates of the constituent curves falling immediately below them.

In phase III. the sliders have advanced through a space (l), and we see that the shape of the string is the same as at first, only in an inverted position.

In phase IV. the sliders have advanced through a space $\frac{3}{2}l$, and the shape of the string is the same as in phase II.

In phase V. each slider has advanced through a space $2l$, and the string has reassumed its original form, so that this phase is merely a repetition of phase I.

The same succession of phases will recur in every subsequent vibration; and the time of each vibration is the time in which one of the sliders, moving with a velocity a, passes over a space $2l$; this time is therefore $\frac{2l}{a}$, as we have before found it to be in a less simple manner.

If the initial form of the string be such as is represented in fig. (2), where M, the middle point between the two fixed extremities A and B, divides the string into two equal segments, one being the exact *inverse* of the other; then constructing as before the constituent forms along A_1D_1 and A_2D_2, we see that the inverted form between B_1C_1 is exactly similar to the

direct forms between A_1B_1 and between C_1D_1: and the same may be said of the forms along A_2D_2. Hence it is plain that the string will have reassumed its initial state when the sliders have advanced only through a space l, which takes place in the time $\frac{l}{a}$; that is, a time half as great as in the former mode of vibration. The second phase here is the middle of the vibration, and the string assumes an inverted position, but it is not the *inverse* of its initial form in the same sense that phase III. is the inverse of phase I. in the former mode of vibration.

It is remarkable, that the middle point M, though fully at liberty to move, remains stationary during the whole time that the string vibrates; for if we take M_1 and M_2, the middle points of the constituent forms, and measure off M_1R_1 and M_2R_2 equal to each other, the former towards D_1 and the latter towards D_2, R_1 and R_2 will arrive under M at the same instant. Now, since the portion of curve $M_1Q_1B_1$ is by the hypothesis the inverse of the portion $B_2Q_2M_2$, the ordinate Q_1R_1 is equal and opposite to Q_2R_2. Hence M will still be in the same place in every position of the two sliders, so that each half of the string vibrates just as if M were a fixed point. This mode of vibration is called the first harmonic, and the sound it produces is the octave above that which is [249]
given by the string in its original mode of vibration.

In the same manner, if the string be initially divided at M and N, see fig. (3), into three equal segments, the segment between MN being the exact inverse of that between MN, and the segment between NB being the exact inverse of that between MN, and therefore a repetition of that between AM, it is easy to see that the string will reassume its initial form when the sliders have moved over the space $\frac{2}{3}\,l$, which will require a time equal to $\frac{2}{3}\,\frac{l}{a}$, or one-third the time of the original vibration. And it may be shown, as in the last case, that the points M and N remain stationary all the time the string vibrates, so that the portions AM, MN, NB vibrate as if each were a separate string. This mode of vibration is called the second harmonic, and the note it gives is the fifth above the note given by the first harmonic, or the twelfth above the fundamental note produced by the original vibration. In the figure three phases are exhibited, the third being a repetition of the first, and the sliders are advanced over a space $\frac{1}{3}\,l$ at each step.

There is no difficulty in extending these considerations to the harmonic sounds of the higher orders, and we shall dismiss this part of the subject with stating, that the third harmonic will give the double octave or fifteenth above the fundamental note, and the fourth will give the interval of a third above that, that is, the interval of a seventeenth above the fundamental note: and it is remarkable, that in the practice of music, these harmonic intervals constitute what is called the common chord, which the ear naturally admits as the most perfect combination of concordant sounds.

It is also worth observing, that these different modes of vibration may all exist at the same time in one and the same string, and may actually be distinguished as existing in the fundamental note by an accurate and musical ear. The mathematical explanation of this depends on a well-known [250] property, which is common to all linear differential equations, or equations in which the differential coefficients of any order appear only under the first degree, being neither multiplied together nor raised to powers. The property consists in this, that if the equations $y = u$, $y = u_1$, $y = u_2$, &c. separately satisfy the proposed linear differential equation, the latter will also be satisfied by making

$$y = u + u_1 + u_2 + \dots .$$

In the present case this conclusion is an immediate consequence of the form of the general integral

$$y = f(x + at) + f(x - at).$$

For if

$$\begin{aligned} y &= \tfrac{1}{2}\,\phi\,(x + at) + \tfrac{1}{2}\,\phi\,(x - at), \\ y &= \tfrac{1}{2}\,\phi_1\,(x + at) + \tfrac{1}{2}\,\phi_1\,(x - at), \\ y &= \tfrac{1}{2}\,\phi_2\,(x + at) + \tfrac{1}{2}\,\phi_2\,(x - at), \\ \&c. &= \&c. \end{aligned}$$

be the integrals which separately correspond to the fundamental sound, the first harmonic, the second harmonic, &c.; and if we take

$$\tfrac{1}{2}\,f(v) = \tfrac{1}{2}\,\phi\,(v) + \tfrac{1}{2}\,\phi_1\,(v) + \tfrac{1}{2}\,\phi_2\,(v) + \dots .$$

then

$$y = \tfrac{1}{2}\,f(x + at) + \tfrac{1}{2}\,f(x - at)$$

will be the integral which corresponds to a vibration in which all the preceding modes of vibration exist at once.

Let us now suppose the initial displacement to extend over a small portion only of the string.

Fig. (4) represents the vibration, in this case, in twelve different phases, the sliders advancing over a space $\frac{1}{6}\,l$ at each step.

Phase I. exhibits the string in its initial position.

In phase II. the original pulse has divided itself into two undulations, one moving towards A and the other towards B, which we will distinguish by the names undulation P and undulation Q.

In phase III. the undulation P is in the act of being reflected at the extremity A of the string, and Q is travelling on with the velocity (a) to the other extremity.

In phase IV. P is completely reflected, and Q has just reached the extremity of the string.

In phase V. P continues its course towards the extremity B, while Q is in the act of being reflected at that extremity.

In phase VI. P is travelling onwards towards B, and is meeting Q, which has just been reflected at B.

In phase VII. P and Q have reunited into one, which is the exact inverse of the original pulse in phase I.; the sliders have now travelled over a space l, and the semivibration is completed.

In phase VIII. the undulations have separated again; P has just reached the extremity B, and Q is travelling towards A.

In phase IX. P is in the act of being reflected at B, [251] and Q continues its course towards A.

In phase X. P has been completely reflected at B, and Q has just reached the extremity A.

In phase XI. P is travelling towards A and Q is in the act of being reflected at A.

In phase XII. P is just meeting Q, which is now completely reflected at A.

In phase XIII. P and Q have again coalesced, and the string is exactly as in phase I. The sliders have now moved over a space $2l$, and the whole vibration has been completed in a time $\frac{2l}{a}$.

Having thus fully discussed the vibrations parallel to y, we need not dwell upon those parallel to z, which will be precisely of the same nature and duration. It will be sufficient to remark, that the vibrations parallel to y may take place simultaneously with those parallel to z, and we may regard the initial form of the string as a curve of double curvature, whose projections on the planes of (y, x) and (z, x) determine the forms of the arbitrary functions. And it is easy to conclude that the string will reassume its original form after the time $\frac{2l}{a}$, this period being common both to the vibrations parallel to y and to those parallel to z.

The longitudinal vibrations parallel to ξ or x, which are given by the equation $\frac{d^2u}{dt^2} = b^2 \left(\frac{d^2u}{dx^2}\right)$, lead to a discussion exactly similar to the preceding. They consist of a series of contractions and dilatations of the string, which travel along it with the uniform velocity b, and suffer reflexion at the ends of the string; and the time which elapses before the string returns to its initial state, or the time of a complete vibration, will be $\frac{2l}{b}$.

There will also be the same series of harmonic sounds, bearing the same relation to the fundamental sound as in the transverse vibrations.

Lastly, all the three species of vibrations, each with its concomitant harmonics, may coexist in the same vibrating string. But the longitudinal vibrations having quite a different period from the transverse, have no musical relation to them, though it is very possible that they may have a material effect in giving a quality to the sound, or in producing what the French term the "timbre" of the note. Experience shows that they produce sounds of a much higher pitch than the transverse vibrations, which requires that b be much greater than a, consequently Q much greater than T.

Remark upon the diagrams.—The ventriculations of the *string*, compared with those of the *sliders*, are in figures (2) and (3) represented too small: the student will have no difficulty in imagining the former increased to about double the size. The same thing occurs in phases I., VII., and XIII. of fig. (4). In the latter figure the pulses occupy a smaller portion of the string than was originally intended, and do not sufficiently taper off at their extremities. The result is that the effect of the *gradual transition* from the direct to the inverted position, as each pulse is echoed at the end of the string, is lost. The intelligent reader will readily supply the deficiency, and need scarcely be reminded that the same method of exhibiting the phases will explain with equal clearness the echoes at the extremities of a vibrating column of air, and the somewhat paradoxical law that a condensed pulse is echoed in the form of a rarefied pulse, and *vice versâ.*

J. P.

[252]

ON A PROPERTY OF CERTAIN PARTIAL DIFFERENTIAL COEFFICIENTS.*

We propose to prove the following proposition—that if R be a function of three rectangular coordinates, x, y, z, such, that all the partial differential coefficients of R of the second order

* From a Correspondent.

are constant and possible, then a system of rectangular axes may always be found, such, that when the coordinates are referred to them, the functions corresponding to $\frac{d^2R}{dx\,dy}\;\frac{d^2R}{dx\,dz}\;\frac{d^2R}{dy\,dz}$ shall all vanish.

We premise the following: that in the case proposed, *any two* of the functions corresponding to $\frac{d^2R}{dx\,dy}\;\frac{d^2R}{dx\,dz}\;\frac{d^2R}{dy\,dz}$ may be made to vanish by transformation of coordinates.

If
$$\left.\begin{aligned} x_1 &= x\cos\theta - y\sin\theta \\ y_1 &= x\sin\theta + y\cos\theta \\ z_1 &= z \end{aligned}\right\}\ldots\ldots\ldots\ldots(a),$$
and
$$\begin{aligned} x' &= x_1, \\ y' &= y_1\cos\phi - z_1\sin\phi, \\ z' &= y_1\sin\phi + z_1\cos\phi, \end{aligned}$$
we shall find that
$$\frac{d^2R}{dx'dz'} = \frac{d^2R}{dx_1dy_1}\sin\phi + \frac{d^2R}{dx_1dz_1}\cos\phi,$$
$$\frac{d^2R}{dy'dz'} = \left(\frac{d^2R}{dy_1^2} - \frac{d^2R}{dz_1^2}\right)\sin\phi\cos\phi + \frac{d^2R}{dy'dz'}\cos 2\phi,$$
so that $\frac{d^2R}{dy'dz'}$, $\frac{d^2R}{dx'dz'}$ both vanish when
$$\tan 2\phi = -\frac{2\dfrac{d^2R}{dy_1dz_1}}{\dfrac{d^2R}{dy_1^2} - \dfrac{d^2R}{dz_1^2}},$$
and
$$\tan\phi = -\frac{\dfrac{d^2R}{dx_1dz_1}}{\dfrac{d^2R}{dx_1dy_1}},$$
in order to which we must have
$$\frac{\dfrac{d^2R}{dy_1dz_1}}{\dfrac{d^2R}{dy_1^2} - \dfrac{d^2R}{dz_1^2}} = \frac{\dfrac{d^2R}{dx_1dz_1}\cdot\dfrac{d^2R}{dx_1dy_1}}{\left(\dfrac{d^2R}{dx_1dy_1}\right)^2 - \left(\dfrac{d^2R}{dx_1dz_1}\right)^2},$$

[253].

$$\text{or}\left(\frac{d^2R}{dx_1dy_1}\cdot\frac{d^2R}{dy_1dz_1}-\frac{d^2R}{dy_1^{\,2}}\,\frac{d^2R}{dx_1dz_1}+\frac{d^2R}{dz_1^{\,3}}\,\frac{d^2R}{dx_1dz_1}\right)\frac{d^2R}{dx_1dy_1}$$
$$-\left(\frac{d^2R}{dx_1dz_1}\right)^2\frac{d^2R}{dy_1dz_1}=0.$$

From equations (a), we get

$$\frac{d^2R}{dy_1dz_1}=\frac{d^2R}{dx\,dz}\sin\theta+\frac{d^2R}{dy\,dz}\cos\theta,$$

$$\frac{d^2R}{dx_1dy_1}=\left(\frac{d^2R}{dx^2}-\frac{d^2R}{dy^2}\right)\sin\theta\cos\theta+\frac{d^2R}{dx\,dy}\cos 2\theta,$$

$$\frac{d^2R}{dx_1dz_1}=\frac{d^2R}{dx\,dz}\cos\theta-\frac{d^2R}{dy\,dz}\sin\theta,$$

$$\frac{d^2R}{dy_1^{\,2}}=\frac{d^2R}{dx^2}\sin^2\theta+\frac{d^2R}{dy^2}\cos^2\theta+\frac{d^2R}{dx\,dy}\sin 2\theta,$$

$$\frac{d^2R}{dz_1^{\,2}}=\frac{d^2R}{dz^2}.$$

By partially substituting these values, our condition may be made to assume the form

$$0=\left.\begin{aligned}&\frac{d^2R}{dx^2}\,\frac{d^2R}{dy\,dz}\sin\theta-\frac{d^2R}{dy^2}\,\frac{d^2R}{dx\,dz}\cos\theta\\&-\left(\frac{d^2R}{dx\,dz}\sin\theta-\frac{d^2R}{dy\,dz}\cos\theta\right)\frac{d^2R}{dx\,dy}\\&+\left(\frac{d^2R}{dx\,dz}\cos\theta-\frac{d^2R}{dy\,dz}\sin\theta\right)\frac{d^2R}{dz^2}\end{aligned}\right\}\frac{d^2R}{dx_1dy_1}-\left(\frac{d^2R}{dx_1dz_1}\right)^2\frac{d^2R}{dy_1dz_1},$$

which evidently is a cubic equation with respect to tan θ. Hence tan θ, and therefore θ, must have at least one real value, and therefore tan ϕ and ϕ must have at least one real value; or the coordinates may be so taken as to cause the functions corresponding to $\frac{d^2R}{dx\,dz}\,\frac{d^2R}{dy\,dz}$ to vanish. Let them be so taken, and let

$$x'=x\cos\omega-y\sin\omega,$$
$$y'=x\sin\omega+y\cos\omega,$$
$$z'=z;$$

then $$\frac{d^2R}{dx'dz'} = \frac{d^2R}{dx\,dz}\cos\omega - \frac{d^2R}{dy\,dz}\sin\omega = 0,$$

$$\frac{d^2R}{dy'dz'} = \frac{d^2R}{dx\,dz}\sin\omega + \frac{d^2R}{dy\,dz}\cos\omega = 0,$$

$$\frac{d^2R}{dx'dy'} = \left(\frac{d^2R}{dx^2} - \frac{d^2R}{dy^2}\right)\sin\omega\cos\omega + \frac{d^2R}{dx\,dy}\cos 2\omega;$$

which last will also vanish if

$$\tan 2\omega = -\frac{2\dfrac{d^2R}{dx\,dy}}{\dfrac{d^2R}{dx^2} - \dfrac{d^2R}{dy^2}}, \qquad [254]$$

a condition which may always be satisfied. Hence the proposition.

The reason why we arrive at a cubic equation for $\tan\theta$ is, obviously, because each one of *the final axes* satisfies the condition of the preliminary proposition.

We might prove, in a similar manner, the existence of the principal axes of bodies.

The above theorem embodies the analytical principle, upon which depends the similarity between the investigations relating to the axes of elasticity, and to diametral planes perpendicular to their ordinates in surfaces of the second order, as those investigations are conducted, respectively, in the first Number of this Journal, and in Hymers's *Analyt. Geom.*, p. 141, 2nd edition. It will be readily seen how our theorem applies to the former of these. With respect to the latter, we would observe, that if

$$R = ax^2 + by^2 + cz^2 + a_1yz + b_1xz + c_1xy + a'x + b'y + c'z + d,$$

$$\frac{d^2R}{dy\,dz} = a_1, \quad \frac{d^2R}{dx\,dz} = b_1, \quad \frac{d^2R}{dx\,dy} = c_1;$$

and hence, by our proposition, the general equation of the second order may, by transformation, be put under the form

$$0 = ax^2 + by^2 + cz^2 + a'x + b'y + c'z + d.$$

Now, if $x = x - \dfrac{a'}{2a}$, this becomes

$$0 = ax^2 + by^2 + cz^2 + b'y + c'z - \frac{a'^2}{4a} + d;$$

hence the plane of yz is a diametral plane perpendicular to

its ordinates. If $a = 0$ the above fails: but as at least one of the quantities a, b, c must be finite, there must be at least one diametral plane perpendicular to its ordinates in every surface of the second order.

M.

NEW METHOD OF SOLVING A BIQUADRATIC EQUATION.

It is known that the method of solving a recurring equation applies equally to equations of the form

$$x^4 + mpx^3 + m^2qx^2 + m^3px + m^4 = 0 \ldots\ldots (1),$$

[255] for by dividing by x^2 and assuming $x + \frac{m^2}{x} = z$, it may be reduced to a quadratic in terms of z. If we represent this equation by

$$x^4 + Px^3 + Qx^2 + Rx + S = 0 \ldots. (2);$$

then, since $P = mp$, $R = m^3p$, and $S = m^4$,

$$P^2S = R^2, \text{ or } S = \frac{R^2}{P^2} \ldots. (3).$$

To determine the corresponding relation between the roots, let $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$, be the roots of the equation

$$x^4 + px^3 + qx^2 + px + 1 = 0,$$

then the roots of equation (1) will be

$$m\alpha, \frac{m}{\alpha}, m\beta, \frac{m}{\beta},$$

and since the products of the 1st and 2nd, and of the 3rd and 4th, are each equal to m^2, these four quantities are proportionals.

Now any biquadratic may be reduced to one whose roots are proportionals; for let a, b, c, d be the roots of any biquadratic, then $a - v, b - v, c - v, d - v$ will be proportionals if

$$(a - v)(d - v) = (b - v)(c - v)$$

or

$$ad - (a + d) v = bc - (b + c) v,$$

or

$$v = \frac{ad - bc}{a + d - (b + c)}.$$

This function admits of only three different values by the interchange of the four letters, and therefore v may be found by a cubic.

Let then $$y^4 + qy^2 + ry + s = 0$$
be any biquadratic equation, deprived of its second term. Assume
$$y = x + v,$$
then the transformed equation is
$$x^4 + 4vx^3 + (6v^2 + q)\,x^2 + (4v^3 + 2qv + r)\,x + v^4 + qv^2 + rv + s = 0 \dots (4).$$

Determine v so that this equation shall be of the form
$$x^4 + Px^3 + Qx^2 + Rx + \frac{R^2}{P^2} = 0 \dots (5);$$
$$\therefore\ v^4 + qv^2 + rv + s = \left(\frac{4v^3 + 2qv + r}{4v}\right)^2,$$
whence by reduction
$$v^3 + \frac{4s - q^2}{2r}\,v^2 - \tfrac{1}{2}qv - \tfrac{1}{8}r = 0.$$

Let a value of v be found from this cubic, and substituted in the coefficients of (4), it will then be of the form (5). It then only remains to divide (5) by x^2, and assume [256]
$$x + \frac{R}{Px} = z,$$
when we shall have
$$z^2 + Pz + Q - \frac{2R}{P} = 0,$$
$$x^2 - zx + \frac{R}{P} = 0,$$
whence four values of x may be found; and then, since
$$y = x + v,$$
the roots of the proposed equation are known.

We have supposed the second term taken away, only to render the coefficients of the reducing cubic simpler than they would otherwise have been; but the method would be exactly the same if the second term had not been taken away.

In this method, as in the others, it may be shown that the reducing cubic is not soluble by Cardan's method, except when the biquadratic has one pair of imaginary roots, and two real roots.

S. S. G.

ON THE EXISTENCE OF A RELATION AMONG THE COEFFICIENTS OF THE EQUATION OF THE SQUARES OF THE DIFFERENCES OF THE ROOTS OF AN EQUATION.*

THE equation of the squares of the differences of the roots gives the means of ascertaining whether any assigned equation has all its roots real; for if they be so, all the roots of the equation of differences must be real and positive, and consequently, by Descartes's rule of signs, all its coefficients must be alternately positive and negative. Accordingly, Waring applied it to this purpose, and in the *Philosophical Transactions* for 1763, gave the conditions of the reality of all the roots in equations of the fourth and fifth degree.

There will be as many conditions as there are coefficients—that is, as there are units in the degree of the equation of the squares of the differences; and therefore, for an equation of the n^{th} order the equation of the squares of the differences
[257] will of course be of the $\frac{n.\overline{n-1}}{2}$ th order. Thus, in the third order there would be three conditions, in the fourth six, and so on.

Lagrange remarked, however, that the number of conditions in these two cases reduced itself to two and three respectively, and he suggested, that a similar simplification might be possible in the ten conditions of the fifth order.

Sturm's theorem, which, however different in form, is still in substance intimately connected with the theory of the equation of the squares of the differences, enables us to ascertain the true number of independent conditions.

By this theorem, we deduce from a given equation $f(x)=0$ a series of n functions. These, with the original $f(x)$, make $n+1$ functions of x. We substitute in them the limits a and b, and the number of changes of sign lost between these limits is the number of real roots of n, which are to be found in this interval. Consequently we have only to write plus and minus infinity in Sturm's functions, to get the whole number of real roots belonging to the equation.

The signs of each function, when $\pm\frac{1}{0}$ is put for x, will of course be that of the first term, supposing each function to be arranged in a series of decreasing powers of x. And if the first term of each be positive, the series of signs at the superior

* From a Correspondent.

limit will be all permanences, and at the inferior all alternations; that is, all the roots of the equation will be real.

Hence the reality of all the roots depends on the signs of $n+1$ terms. But of these, the sign of $f(x)$ is determined at the limits $\pm\frac{1}{0}$; so is that of $\frac{df(x)}{dx}$, which is Sturm's first function. Consequently there remain but $n-1$ terms on the sign of which the reality of roots depends. Instead, therefore, of $\frac{n.\overline{n-1}}{2}$ conditions, there are in reality but $n-1$.

Thus, in the equation of the third degree we find two conditions, in that of the fourth three, and so on, agreeing with what Lagrange found in these cases, and suspected in that of the fifth degree.

It is not very difficult to see why some of the coefficients of the equations of differences must be so connected with the rest, as not to give any independent condition.

In order to get an idea of this connection, let us imagine $n-1$ independent conditions, that is, $n-1$ functions of the coefficients of $f(x)=0$, which have a definite sign when all the roots are real. These functions are coefficients of the equation of the squares of the differences. Let them all become equal to zero, then we have $n-1$ relations [258] among the n roots, which will give every one of these roots, except one, in terms of that root. Now, the relations $b=a$, $c=a,\ldots k=a$, will fulfil our $n-1$ equations, because these evidently make all the coefficients of the equation of the squares of the differences vanish. Hence these are the relations implied in the $n-1$ equations we have assumed; and these would, it has just been said, make all the coefficients $=0$. Consequently, $n-1$ independent relations are all we can have, and it follows, that if $n-1$ independent coefficients of the equation of differences become $=0$, all will be so.

We may take the matter somewhat differently, still using the case of equal roots in $f(x)=0$ to show the relations among the coefficients of the equation of the squares of the differences.

If $f(x)=0$ has m equal roots, there will be $\frac{m(m-1)}{2}$ roots of the equation of the squares of the differences, or $\Delta=0$, equal to zero. Let $f(x)=0$ get another root equal to these m roots by any change in its coefficients, then there will be $\frac{(m+1)m}{2}$ roots in $\Delta=0$ equal to zero; the difference is m.

Thus, a single fresh relation among the coefficients of $f(x) = 0$ makes m coefficients of $\Delta = 0$ vanish; for obviously the last coefficients of this equation disappear whenever it gets roots equal to zero.

We may easily see, too, that the constant function (the last in Sturm's process) is the same as the term independent of u in $\Delta = 0$.

The equation $\Delta = 0$ may, theoretically, be got by eliminating x between the two equations

$$f(x) = 0 \text{ and } f'x + \frac{u}{2} f''x + \frac{u^2}{2.3} f'''x + \ldots u^{n-1} = 0,$$

(*Lagrange*, p. 7); Δ will be the term independent of x: put, then, $u = 0$, Δ reduces itself to its last term, and the process becomes simply that of finding the common measure of $f(x)$ and $f'(x)$, which, abstracting the changes of sign, is exactly Sturm's process; hence his term independent of x, will be "aux signes près" the constant term in $\Delta = 0$.

The development of this idea would undoubtedly lead to the general theory of Sturm's method, and would make it more than a happy artifice, by showing its intimate connection with the equation of the squares of the differences. As is generally the case, the different ways in which the subject may be viewed, ultimately coalesce.

R. L. E.

[259]

ON THE EXISTENCE OF BRANCHES OF CURVES IN SEVERAL PLANES.

In tracing a curve expressed by an equation between two variables, it is customary to make use of negative as well as of positive values of the variables, but to reject those which are usually called impossible or imaginary. This practice was allowable so long as it was supposed that impossible quantities had no meaning in geometry; but if we once admit the possibility of interpreting them in this science, though not in arithmetic, we are bound in strict logic not to neglect them. Accordingly, the Abbé Buée, in his very ingenious paper in the *Philosophical Transactions* for 1806, in which he demonstrated the possibility of interpreting geometrically the symbol usually written $\sqrt{(-1)}$, showed that quantities affected by it corresponded to branches of the curve situate in a plane at right angles to the original plane. Professor Peacock is,

I believe, the only author who agrees with Buée in this view of the nature of curves, although it seems difficult to show any reason why it should not be generally allowed. I believe that a name has had great influence in preventing its adoption, the word *imaginary* so frequently applied to the symbol $\sqrt{(-1)}$ appearing to make persons unwilling to believe that it could possibly admit of any interpretation. Yet, after all, the difference between it and the symbol $-$ is not so very great, both admitting of easy interpretation in the science of geometry, and neither, if considered independently, in the science of arithmetic, more especially if we consider them, as I have done elsewhere, as fractional powers of the symbol $+$, having peculiar properties depending on the fundamental definition of that symbol.

It appears to me, that if we once admit anything beyond what are called positive values of the variables, that is, pure arithmetical values wholly independent of the symbol $+$, there is no reason why we should confine ourselves to $-$ or $+^{\frac{1}{2}}$, since this is not differently circumstanced from any other power of $+$, analytically considered. I therefore hold, that we must either limit ourselves to the one quadrant formed by the positive axes, or we must be prepared to consider the curve as existing in several planes. Nor need it appear surprising, that by means of an equation between two variables, we are able to take into our view three dimensions, for the symbol $+$ is really equivalent to an angular coordinate, and therefore enables us to reach all points of space. In the following pages I propose, therefore, to extend farther than he has done, the principle introduced by Buée, and, not confining myself to values of the variables of the form $+^{\frac{1}{4}}a$, to investigate the forms of curves, when we assume that the variables may be of the general form $+^{\frac{p}{q}}\ a$.

As a preliminary, we must consider what is the meaning of this expression when substituted in the equation to a curve $y = f(x)$. Since $+^{\frac{p}{q}}$ represents the turning of a line through an angle $\dfrac{2p\pi}{q}$, the expression $x = +^{\frac{p}{q}}a$ signifies that we are to measure a line whose length is a, along the axis of x, and then to turn the axis through an angle equal to $\dfrac{2p\pi}{q}$. But we may turn the axis in an infinite number of ways, which at first would make it appear that $x = +^{\frac{p}{q}}a$ would [260]

not give a definite point. But it is to be observed, that the axis of x is always to be perpendicular to that of y, so that it is only allowable to move the axis in a plane perpendicular to the plane of xy. If the substitution of $+^{\frac{p}{q}}a$ for x gives a value of y equal to $+^{\frac{r}{s}}b$, this implies that we have to measure a length b along y, and then to turn it through an angle equal to $\frac{2r\pi}{s}$, and the plane passing through the two new axes will be the plane of a branch of the curve formed by assigning all values to a from 0 to ∞, unless, which will not unfrequently happen, the value of a affects the index of $+$ in the expression for y, in which case every element of the curve will lie in a different plane from the contiguous one, and the curve will be one of double curvature. This perhaps will appear more clearly if we illustrate it by an example in the case of the parabola, which is the simplest curve for our purpose. The equation to the parabola being

$$y^2 = mx,$$

if we put $+^{\frac{p}{q}}a$ for x, we find the value of y to be $+^{\frac{p}{2q}}m^{\frac{1}{2}}x^{\frac{1}{2}}$. Hence we see that the axis of y is to be turned round through an angle, which is one-half of that through which x is turned round; and for all the values which we may assign to a, if we leave p and q unchanged, the plane of the branch will remain unchanged, and the branch itself will be exactly similar to that in the plane xy, since the *numerical* relation between y and x is the same, whatever values we assign to p and q. By changing these last we obtain different branches in different planes; and as there is no limit to the values we may assign to them, it appears that the equation to the parabola, considered generally, represents a curve of an infinite number of branches, all passing through the origin, and situate in planes, such that the axis of x in any plane makes with the old axis twice the angle which the axis of y in that plane does with the old axis of y. As a particular case, we may take $\frac{p}{q} = \frac{1}{2}$, or make $x = -a$, whence

[261] $$y = +^{\frac{1}{4}}(ma)^{\frac{1}{2}},$$

or the curve lies in a plane at right angles to the old plane.

The existence of this curve on the negative side of the axis of y, serves to explain an apparent anomaly which occurs in

a very elementary problem. If we seek the equation to the locus of the intersection of a perpendicular from the focus of a parabola on the tangent, we obtain an equation which divides itself into two—the one representing the axis of y, which is the solution usually taken, the other furnishing only the focus. It seems strange that the focus should in any way be a solution of the problem, since the tangent of the positive branch of the curve never passes through that point. But if we consider the branch of the curve which lies in a plane perpendicular to the plane of xy, we see that all the tangents to that branch must pass through the positive axes of x, and consequently that one, or rather two, must pass through the focus, which thus is the point in which the tangent is met by a perpendicular from that point. Moreover, we find that the values of x and y, which belong to the focus, render the equation to the tangent of the form

$$y = (-)^{\frac{1}{2}} x + \beta,$$

showing that the tangent is in a plane at right angles to its original plane.

The form of the equation to the parabola renders it very easy to determine the value of y corresponding to that of x; but in the other curves of the second degree, though the investigation may not be so simple, we arrive at similar conclusions. Taking the equation to the ellipse referred to the centre,

$$y^2 = m^2 (a^2 - x^2),$$

$\left(\text{putting } \frac{b^2}{a^2} = m^2\right)$, we have for the value of y

$$y = m (a^2 - x^2)^{\frac{1}{2}}.$$

So long as $x < a$, whether we reckon x to be positive or negative, the value of y is possible, and the curve exists only in the plane of xy. If we make $x > a$, x being either positive or negative, we find y to be of the form $-^{\frac{1}{2}} mp$, showing that there is a branch of the curve in a plane at right angles to that of xy. Its form, it will be easily seen, is that of a hyperbola, since y increases with x, and becomes infinite when x is so, the relation between them being of the form

$$y = m (x^2 + a^2)^{\frac{1}{2}},$$

which is the equation to a hyperbola.

Hence, the vertices of the major axes of the ellipse are the vertices of two hyperbolas in a plane at right angles to the plane of the ellipse, and the ratio of the axes of which is the same as that of the axes of the ellipse. Also, since x and y are symmetrically involved in the equation to the ellipse,

[262] there must be a similar result for the extremity of the minor axis, the only difference being, that the axes of the hyperbola will be reversed in position.

If, more generally, we suppose $x = +^{\frac{p}{q}} c$, the result is not so simple, for we have

$$y^2 = m^2 (a^2 - +^{\frac{2p}{q}} c^2);$$

from which we cannot directly determine the angle through which the axis of y is turned, corresponding to that through which x is supposed to be turned; but if we avail ourselves of the connexion which subsists between powers of + and Demoivre's formula, we are able to determine the value of y.

Let $+^{\frac{p}{q}} = \cos\theta + -^{\frac{1}{2}} \sin\theta$, so that $\theta = \dfrac{2p\pi}{q}$,

then $$y^2 = m^2 \{a^2 - (\cos 2\theta + -^{\frac{1}{2}} \sin 2\theta)\, c^2\},$$
$$= m^2 \{a^2 - c^2 \cos 2\theta - -^{\frac{1}{2}} c^2 \sin 2\theta\};$$

whence $y = m\,(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{4}} (\cos\phi - -^{\frac{1}{2}} \sin\phi)$,

where $$\cos 2\phi = \frac{a^2 - c^2 \cos 2\theta}{(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}},$$
$$\sin 2\phi = \frac{c^2 \sin 2\theta}{(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}}.$$

This result differs materially from that in the case of the parabola; for since the value of ϕ depends on c, the circulating function $\cos\phi - -^{\frac{1}{2}} \sin\phi$ depends on c, so that the angle through which the axis of y is to be turned, varies with the length of the abscissa, and the plane of the curve is constantly changing for every value of x, or in other words, the curve is one of double curvature.

When $c = 0$, $y = b$, and the curve passes through the extremities of the axis minor; and since c may be made infinite, the curve is infinite. Also, since x and y are symmetrically involved in the equation, there will be a similar curve passing through the extremities of the axis major. Hence, in addition to the hyperbolas already mentioned, the equation to the ellipse includes an infinite number of curves, with infinite branches passing through the extremities of the axes.

It is not necessary to consider the equation to the hyperbola, since it evidently leads to similar results. Nor is there much interest attached to the discussion of other more complicated curves, which I shall therefore omit with these

exceptions—the curves of sines and cosines, and the logarithmic curve. These I shall briefly touch on, as the form of their equations renders their discussion easy, and as there is considerable interest attached to them in a geometrical point of view.

If $$y = a \sin x,$$

then, making $x = +^{\frac{p}{q}} c$, we have [263]

$$y = a \sin (+^{\frac{p}{q}} c);$$

and it remains to be considered what relation this bears to $\sin c$. If we suppose the sector of the circle whose angle is c to turn round the radius from which c is measured, it will be easily seen, that on turning it round through a circumference the angle will return to its original position, and so for any number of revolutions. Therefore, this operation of turning the sector through a circumference is subject to the laws of the symbol $+$, and may therefore be represented by it; and consequently, the operation of turning the sector through the $\frac{p}{q}$th part of a circumference will be properly represented by $+^{\frac{p}{q}} c$. But since the sine of the arc, being in the plane of the sector, is perpendicular to the axis of revolution, it will also be moved through the $\frac{p}{q}$th of a circumference, from which it follows that

$$\sin (+^{\frac{p}{q}} c) = +^{\frac{p}{q}} \sin c.$$

But the cosine being measured along the axis of revolution, experiences no change corresponding to the change in the angle, so that we have

$$\cos (+^{\frac{p}{q}} c) = \cos c.$$

It may be observed, that these propositions are extensions of the two common theorems, that

$$\sin (-x) = -\sin x, \quad \cos (-x) = \cos x.$$

Hence, in the case of the curve of sines, we have

$$y = +^{\frac{p}{q}} a \sin c,$$

which shows that the axis of y is to be turned through an angle equal to that through which the axis of x is turned. Hence, if we suppose the plane of xy to turn round an axis

in its own plane, passing through the origin, and making equal angles with the two axes of x and y, in every position there will be a curve of sines exactly the same as that in the plane of xy. On the other hand, the equation to the curve of cosines being

$$y = a \cos x,$$

gives for $x = +^{\frac{p}{q}} c$,

$$y = a \cos (+^{\frac{p}{q}} c) = a \cos c,$$

which shows that the axis of y remains fixed: and if we suppose the plane of xy to turn round the axis of y, there will be [264] a curve of cosines, the same as that in xy, corresponding to every position of the plane.

The equation to the logarithmic curve is

$$y = \varepsilon^{nx}.$$

Let now $x = +^{\frac{p}{q}} a = a (\cos \theta + -^{\frac{1}{2}} \sin \theta)$ suppose.

Then $$y = \varepsilon^{na (\cos \theta + -^{\frac{1}{2}} \sin \theta)} = \varepsilon^{na \cos \theta} . \varepsilon^{na \sin \theta \, -^{\frac{1}{2}}}.$$

Now, since $+^{r} = \varepsilon^{2r\pi -^{\frac{1}{2}}}$, this gives us

$$y = (+)^{\frac{na \sin \theta}{2\pi}} . \varepsilon^{na \cos \theta};$$

thus determining the angle through which the axis of y is to be turned: and as this depends on a, the angle must (as in the case of the ellipse) vary with the length of the abscissa, so that the curve is not situate in one plane, but is a curve of double curvature. The absolute linear value of y, it will be easily seen, is less than that corresponding to the same linear value of x in the plane of xy, since the index of ε is reduced in the ratio of the cosine of θ to unity. It seems scarcely worth while further to discuss the nature of this curve, but having here adopted very different ideas concerning it from those promulgated by M. Vincent in Gergonne's *Annales des Mathématiques,* I think it necessary to state more at length my reasons for differing from that author.

In a paper in the preceding number of this Journal, I developed what I conceive to be the true theory of general logarithms, and I endeavoured to show that the impossible parts are really logarithms of the powers of +, the existence of which is generally overlooked. I pointed out that the formula of Mr. Graves, which agrees with that of M. Vincent, was derived from the supposition, that the base of the system of logarithms was of the form $+^{r} a$. This gives for the equation to the logarithmic curve,

$$y = (+^{r} a)^{x}.$$

Now M. Vincent assumes, that when x is fractional, y has as many values as there are units in the denominator of x; and when that is even, that two of these are possible—one positive and the other negative. Then he shows that the latter values do not form a continuous curve, but one of a kind which he calls *ponctuées*, whose nature is very peculiar, since, though the points are infinitely near to each other, yet we are able to draw an infinite number of straight lines between any two. This very strange result, so contrary to all our preconceived ideas of the nature of a curve, is sufficient, I think, to make us doubt the correctness of the method: but it is not very easy to point out the error, unless we employ the mode of considering the origin of the plurality of roots, which I have explained in the paper above referred to. According to that system, these various roots arise from our supposing a change to be made in the value of r; and properly speaking [265] there is no plurality of roots, but the nature of the quantity $+^r a$, whose root is taken, is indeterminate. Now here, in the case of the equation to the logarithmic curve, there is no indeterminateness, since r can only have one value. Any change in the value of r is a change in one of the constants of the equation to the curve, and consequently the equation no longer represents the same curve. Each of the points of the "*courbe ponctuée*" of M. Vincent, is really the point in which a certain curve meets the plane of xy. They are therefore wholly unconnected with each other, and cannot be reckoned as belonging to the same curve, either in a geometrical or analytical sense. This being granted, we perceive that there is no foundation for M. Vincent's very anomalous conclusion; and we are thus relieved from the necessity of believing in the existence of a species of line, of which we can hardly form a conception, and which has no sort of analogy to support it.

I said, that each point in the "*courbe ponctuée*" of M. Vincent belonged to a separate curve—it may be interesting to consider for a moment its nature. The equation

$$y = (+^r a)^x,$$

gives

$$y = +^{rx} a^x.$$

So long as x is an integer, this is the same as the simple equation

$$y = a^x,$$

and gives us the logarithmic curve in the plane of xy: but if we suppose x to be a fraction, it appears that the axis of y is to be turned through rx^{th} part of a circumference. As this

angle varies with x, it appears that the curve is one of double curvature, and, as when $x = 0$, $y = 1$, it intersects the plane of xy at a distance 1 along the axis of y. But it may intersect it again; for if

$$x = \frac{1}{2r}, +^{rx} = +^{\frac{1}{2}} = -,$$

and it cuts the plane of xy on the negative side of y; and if $x = \frac{1}{r}$, then $+^{rx} = +$, and it cuts the plane of xy on the positive side of y, meeting the curve traced in the plane of xy. On increasing x we shall obtain a similar curve which cuts the plane of xy, first above and then below the axis of x, the curve meeting the plane of xy whenever $rx = \frac{m}{2}$, m being either odd or even. It is needless to enter into the discussion of the complicated cases when x is of the form $+^{\frac{p}{q}} a$; more particularly as this paper has already exceeded its just limits. And I only will add, that these speculations derive their chief value from their bearing on the General Theory of the Science of Symbols. Practically, little attention will be paid to curves existing out of the plane of reference, since [266] the curves themselves do not come sufficiently under our eye to attract much interest. Perhaps the only way in which the existence of such curves is likely to be brought into notice, is in those cases where they serve to show the possibility of the interpretation of a solution of an equation. One case of this kind I have remarked on in this paper; several have been pointed out by Buée, and many I have no doubt will be added, when the attention of mathematicians is more particularly directed to the subject.

ON A PROPERTY OF THE HYPERBOLA.*

In a paper published in the *Philosophical Transactions* for 1836, the author, H. F. Talbot, Esq., has given a demonstration of a new theorem, connecting three arcs of the equilateral hyperbola. It may be stated thus: If in the equilateral hyperbola, referred to its asymptotes as axes, the sum of three abscissæ be zero, their product r, and the sum of the

* From a Correspondent.

products of every pair $-\dfrac{r^2}{4}$, the sum of the arcs subtended by those abscissæ $= \frac{3}{4}r +$ const. The object of the present communication is, to prove that this theorem is included in a more general one, which gives a relation between three arcs of *any* hyperbola, referred to its asymptotes. The investigation furnishes a good example of Mr. Talbot's general method of finding the sum of a series of integrals, and may perhaps tempt some to consult the original memoir. There is another on the same subject in the *Transactions* for 1837, Part I.

The equation to the hyperbola, referred to its asymptotes, is

$$xy = \tfrac{1}{2}(a^2 + b^2);$$

and if a be the angle between the asymptotes,

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2 + 2\cos a\,\frac{dy}{dx}\right\}},$$

$$= \frac{1}{x^2}\sqrt{(x^4 - 2m^2x^2\cos a + m^4)},$$

where $\qquad m^2 = \tfrac{1}{2}(a^2 + b^2)$.

Hence the arc $\quad = \displaystyle\int\frac{dx}{x^2}\sqrt{(x^4 - 2m^2x^2\cos a + m^4)}$.

Now, assume

$$\sqrt{(x^4 - 2m^2x^2\cos a + m^4)} = vx + m^2,$$

where v is a variable quantity. Squaring and dividing [267] by x, we have

$$x^3 - (2m^2\cos a + v^2)\,x - 2m^2v = 0.$$

Let x, y, z be the roots of this equation, so that

$$x + y + z = 0,\; xy + xz + yz = -(2m^2\cos a + v^2),\; xyz = 2m^2v.$$

Then
$$\sqrt{(x^4 - 2m^2x^2\cos a + m^4)} = vx + m^2,$$
$$\sqrt{(y^4 - 2m^2y^2\cos a + m^4)} = vy + m^2,$$
$$\sqrt{(z^4 - 2m^2z^2\cos a + m^4)} = vz + m^2.$$

Multiply these equations by $\dfrac{dx}{x^2}, \dfrac{dy}{y^2}, \dfrac{dz}{z^2}$, respectively, and add; then, denoting

$$\frac{1}{x^2}\sqrt{(x^4 - 2m^2x^2\cos a + m^4)} \text{ by } \phi x,$$

$$\phi x\,.\,dx + \phi y\,.\,dy + \phi z\,.\,dz = v\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right)$$
$$+ m^2\left(\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}\right).$$

But $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = d \,.\, \log(xyz),$$
$$= d \,.\, \log 2m^2 v,$$
$$= \frac{dv}{v},$$
$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = -d \,.\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right),$$
$$= -d \,.\, \frac{xy + xz + yz}{xyz},$$
$$= d \,.\, \frac{2m^2 \cos a + v^2}{2m^2 v},$$
$$= \frac{dv}{2m^2} + \cos a d \,.\, \frac{1}{v};$$

therefore $\phi x \,.\, dx + \phi y \,.\, dy + \phi z \,.\, dz = \frac{3}{2}\, dv + m^2 \cos a \,.\, d \,.\, \frac{1}{v}$.

Integrating, we have
$$\int \phi x \,.\, dx + \int \phi y \,.\, dy + \int \phi z \,.\, dz = \frac{3}{2}\, v + \frac{m^2 \cos a}{v} + \text{const.}$$

Let the product of the abscissæ be r, which gives $v = \frac{r}{2m^3}$. Substituting for v this value, and for m^2 and $\cos a$ their values $\frac{1}{2}(a^2 + b^2)$ and $\frac{a^2 - b^2}{a^2 + b^2}$,

[268]

$$\int \phi x \,.\, dx + \int \phi y \,.\, dy + \int \phi z \,.\, dz = \frac{3}{2} \,.\, \frac{r}{a^2 + b^2} + \frac{1}{2} \,.\, \frac{a^4 - b^4}{r} + \text{const.}$$

The equations of condition become
$$x + y + z = 0, \quad \text{and } xy + xz + yz = -(a^2 - b^2) - \frac{r^2}{(a^2 + b^2)^2}.$$

The theorem, therefore, may be stated thus: If in any hyperbola, referred to its asymptotes, the sum of three abscissæ be zero, their product r, and the sum of the products of every pair,
$$-(a^2 - b^2) - \frac{r^2}{(a^2 + b^2)^2},$$
then the sum of the corresponding arcs
$$= \frac{3}{2} \,.\, \frac{r}{a^2 + b^2} + \frac{1}{2} \,.\, \frac{a^4 - b^4}{r} + \text{const.}$$

If we make $a = b = 1$, this reduces itself to Talbot's theorem.

It may be interesting to try a numerical example. For this purpose, integrating between the limits (x, x') of x, (y, y') of y, (z, z') of z, we have

$$\int_x^{x'} \phi x \,.\, dx + \int_y^{y'} \phi y \,.\, dy + \int_z^{z'} \phi z \,.\, dz = \tfrac{3}{2} \,.\, \frac{r' - r}{a^2 + b^2} + \frac{a^4 - b^4}{2}\left(\frac{1}{r'} - \frac{1}{r}\right).$$

Let $$a = \sqrt{2}, \quad b = 1;$$

in which case the equation for determining x, y, z becomes

$$x^3 - (1 + v^2)\, x - 3v = 0.$$

This is satisfied by the following values:

$$x = 1, \quad y = 2.5414, \quad z = -3.5414, \quad r = 3v = -9,$$

$$x' = 1.05, \quad y' = 2.3974, \quad z' = -3.4474, \quad r' = -8.6778.$$

Approximating to the value of the arc subtended by the portion of the abscissæ $x' - x$, and calling it arc (x), we find

$$\text{arc}\,(x) = .0721.$$

And similarly, $$\text{arc}\,(y) = .1364,$$

$$\text{arc}\,(z) = .0907.$$

Now, for a reason which we will state presently, arc (x) must be considered negative. Therefore we have

$$\begin{aligned} \text{arc}\,(y) + \text{arc}\,(z) &= .2271 \\ -\,\text{arc}\,(x) &= .0721 \\ \hline \text{sum} &= .1550 \end{aligned}$$

Also, $$\tfrac{3}{2} \,.\, \frac{r' - r}{a^2 + b^2} = \tfrac{1}{2}(r' - r) = \quad .1611$$

$$\frac{a^4 - b^4}{2}\left(\frac{1}{r'} - \frac{1}{r}\right) = \tfrac{3}{2}\left(\frac{1}{r'} - \frac{1}{r}\right) = -\,.0061$$

$$\text{sum} = \quad .1550$$

which agrees with the theorem.

Mr. Talbot's reasoning with regard to the signs of [269]
the arcs is of this kind. Taking the example before us, since x and y are both positive and z negative, vx or $\dfrac{x^2yz}{3}$ is negative; therefore its equivalent

$$\sqrt{(x^4 - 2m^2x^2 \cos a + m^4)} - m^2 \quad \text{or} \quad \sqrt{(x^4 - x^2 + \tfrac{9}{4})} - \tfrac{3}{2}$$

must be negative; hence, since the values of x are between 1 and 1.05, the radical must have a negative sign. Similar reasoning will shew that $\sqrt{(y^4 - y^2 + \tfrac{9}{4})}$ must have a negative,

and $\sqrt{(z^4 - z^2 + \frac{9}{4})}$ a positive sign. Therefore the three integrals must be written

$$-\int \frac{dx}{x^2} \sqrt{(x^4 - x^2 + \tfrac{9}{4})}, \ -\int \frac{dy}{y^2} \sqrt{(y^4 - y^2 + \tfrac{9}{4})}, \ +\int \frac{dz}{z^2} \sqrt{(z^4 - z^2 + \tfrac{9}{4})}.$$

Finally, since y' is less than y,

$$-\int_y^{y'} \frac{dy}{y^2} \sqrt{(y^4 - y^2 + \tfrac{9}{4})}$$

is positive. Hence the first arc must be considered negative, and the second and third positive.

Q.

ON THE ACHROMATISM OF EYE-PIECES OF TELESCOPES AND MICROSCOPES.*

MR. AIRY, in a paper published in the Second Volume of the *Cambridge Transactions*, has investigated the conditions under which a system of lenses is achromatic, *per se*—that is, when only one kind of glass is made use of. The enquiry, on account of its immediate application to the eye-pieces of telescopes and microscopes, is one of considerable importance: and as the way in which it is conducted in the paper just mentioned appears to be more complicated than necessary, and does not lead to the most general solution of the problem, perhaps the following attempt may be not wholly without interest.

The difficulty of the question consists in finding the angle which a ray of light makes with the axis of the system of lenses after having passed through it. When this is done, we have only to take the chromatic variation, and equate it to zero, to get the general equation of achromatism for the system.

In what follows, lines are considered positive when measured in a direction opposite to that of the incident ray.

Let y_n be the tangent of the angle which the ray makes
[270] with the axis before its incidence on the n^{th} lens of the system; let z_n be the distance from the axis of the point where it impinges on that lens; and take a_{n-1} to signify the distance of the n^{th} from the $n - 1^{\text{th}}$ lens. Then $\frac{z_n}{y_n}$ and $\frac{z_n}{y_{n+1}}$

* From a Correspondent.

are the distances of the conjugate foci from the n^{th} lens, and by the ordinary formula

$$\frac{y_{n+1}}{z_n} - \frac{y_n}{z_n} = \rho_n,$$

ρ being the reciprocal of the focal length; therefore

$$y_{n+1} - y_n = \rho_n z_n.$$

Again, we have the simply geometrical relation

$$\frac{z_{n+1}}{y_{n+1}} - \frac{z_n}{y_{n+1}} = a_n,$$

or

$$z_{n+1} - z_n = a_n y_{n+1}.$$

The advantage gained by introducing y_n is, that we thus have precisely similar equations for the optical and geometrical conditions of the problem. Nothing is now easier than by successive substitutions to determine the value of y_n in terms of y and z, and y_n is the tangent of the "visual angle," which we are seeking. As an instance, let it be proposed to determine the conditions of achromatism in a system of three lenses. Mr. Airy has done this only for rays originally parallel to the axis: the method here proposed applies with equal facility to the general case.

The equations required are these,

$$\begin{aligned} y_2 - y_1 &= \rho_1 z_1 \\ z_2 - z_1 &= a_1 y_2 \\ y_3 - y_2 &= \rho_2 z_2 \\ z_3 - z_2 &= a_2 y_3 \\ y_4 - y_3 &= \rho_3 z_3. \end{aligned}$$

Hence

$$y_2 = y_1 + \rho_1 z_1$$

$$z_2 = (1 + a_1\rho_1)\, z_1 + a_1 y_1$$

$$y_3 = [1 + a_1\rho_2]\, y_1 + [\rho_1 + \rho_2 + a_1\rho_1\rho_2]\, z_1$$

$$z_3 = (1 + a_1\rho_1 + a_2\rho_1 + a_2\rho_2 + a_1a_2\rho_1\rho_2)\, z_1 + (a_1 + a_2 + a_1a_2\rho_2)\, y_1$$

$$y_4 = [1 + a_1\rho_2 + a_1\rho_3 + a_2\rho_3 + a_1a_2\rho_2\rho_3]\, y_1 + [\rho_1 + \rho_2 + \rho_3 + a_1\rho_1\rho_2 + a_1\rho_1\rho_3 + a_2\rho_1\rho_3 + a_2\rho_2\rho_3 + a_1a_2\rho_1\rho_2\rho_3]\, z_1.$$

Taking the chromatic variations of the two terms in the usual way, and equating each to zero, we find

$$a_1\rho_2 + a_1\rho_3 + a_2\rho_3 + 2a_1a_2\rho_2\rho_3 = 0$$

$$\rho_1 + \rho_2 + \rho_3 + 2a_1\rho_1\rho_2 + 2a_1\rho_1\rho_3 + 2a_2\rho_1\rho_3 + 2a_2\rho_2\rho_3 + 3a_1a_2\rho_1\rho_2\rho_3 = 0.$$

The first of these equations becomes unnecessary in the particular case considered by Mr. Airy, viz. that in which

[271] $y_1 = 0$; the second is identical with that given by him at p. 245, when attention is paid to the signs. Taken together they determine the relative positions of three given lenses, which shall form a combination achromatic for rays of any degree of obliquity.

In the particular case in which the focal distances of all the lenses are equal, and the intervals a_1, a_2, &c. are also equal, the general equations degenerate into a system of simultaneous equations in finite differences. They are then

$$y_{n+1} - y_n = \rho z_n, \quad z_{n+1} - z_n = a y_{n+1}.$$

Eliminating z_n, we get

$$y_{n+2} - (\rho a + 2)\, y_{n+1} + y_n = 0.$$

The general solution of this will be

$$y_n = cA^n + c_1 A^{-n},$$

A being a root of the recurring quadratic equation

$$x^2 - (\rho a + 2)\, x + 1 = 0,$$

c and c are to be found by the conditions

$$y_1 = cA + c_1 A^{-1}, \quad y_1 + \rho z_1 = cA^2 + c_1 A^{-2}.$$

The general solution of the system of quasi-equations employed in the enquiry must involve some functional operation which degenerates into the radical contained in A.

It would be perhaps worth considering how far we might be able to present this operation in a distinct form, defined and distinguished by a particular symbol; but the subject is not one which can be discussed at present. At any rate, we see that the research of the general expression for y_n is one of considerable difficulty.

The greater part of the investigation given by Mr. Airy in the conclusion of his paper, with respect to the achromatism of microscopes, becomes unnecessary by employing the general expression given above for y_4. His object is to determine the distance of an object-glass of given focal length from a diaphragm whose distance from the field-glass of a given eye-piece of three lenses is given.

Let a_0 be the distance of the diaphragm from the field-glass; therefore we have $z_1 = a_0 y_1$, and putting this value for z_1, we get an expression for y_4 of the form $y_4 = y_1 R$. The chromatic variation of this is to be zero, and consequently that of its logarithm;

$$\therefore 0 = \frac{\Delta y_1}{y_1} + \frac{\Delta R}{R}.$$

Now Mr. Airy has shown that (adopting the notation of this paper)

$$\frac{\Delta y_1}{y_1} = - x\rho_0 \frac{\delta\mu}{\mu - 1},$$

x being the distance of the object-glass from the diaphragm, and ρ_0 its vergency, or the reciprocal of its focal length. Putting for R its value, we get at once $x\rho_0 =$

$$\begin{aligned}[a_0 [\rho_1 + \rho_2 + \rho_3] + a_1 [\rho_2 + \rho_3] + a_2\rho_3 + 2a_0a_1 [\rho_1\rho_2 + \rho_1\rho_3] \quad [272] \\ + 2a_0a_2 [\rho_1\rho_3 + \rho_2\rho_3] + 2a_1a_2\rho_2\rho_3 + 3a_0a_1a_2\rho_1\rho_2\rho_3]\end{aligned}$$

divided by

$$\begin{aligned}[1 + a [\rho_1 + \rho_2 + \rho_3] + a_1 [\rho_2 + \rho_3] + a_2\rho_3 + a_0a_1 [\rho_1\rho_2 + \rho_1\rho_3] \\ + a_1a_2 [\rho_1\rho_3 + \rho_2\rho_3] + a_1a_2\rho_2\rho_3 + a_0a_1a_2\rho_1\rho_2\rho_3]\end{aligned}$$

which is identical with his result.

R. L. E.

ON THE TRANSFORMATION OF HOMOGENEOUS FUNCTIONS OF THE SECOND DEGREE.

(1) A homogeneous function, of the second degree, of any number of variables, may always be transformed into another which shall contain only the squares of the new variables. The method of doing this will be found in a paper by M. Lebesgue, in *Liouville's Journal de Mathématiques*, tom. II. p. 337. The case in which the variables are three in number, and the transformation amounts to a change in the axes of coordinates, frequently occurs. It presents itself in the reduction of the equation to surfaces of the second order; and the properties of principal axes of rotation, of the axes of elasticity of a crystal, &c. may be shown to depend upon the same transformation.

Thus the moment of inertia of a solid about an axis whose direction-cosines are $l, m, n,$ is

$$\begin{aligned}= \int(x^2 + y^2 + z^2)\, dM - l^2 \int x^2\, dM - m^2 \int y^2\, dM - n^2 \int z^2\, dM \\ - 2mn \int yz\, dM - 2ln \int xz\, dM - 2lm \int xy\, dM. \ldots (1).\end{aligned}$$

The form of the body being known, the integrals are given quantities, which we may write A, B, C, f, g, h.

The first term is independent of the direction of the axes. The remainder of the expression, changing the signs, is

$$Al^2 + Bm^2 + Cn^2 + 2fmn + 2gln + 2hlm. \ldots (2).$$

If x_1, y_1, z_1 be the coordinates of a point referred to new axes, l_1, m_1, n_1 the direction-cosines of the axis of inertia, then, by the known formulæ,

$$\left.\begin{array}{l} x = ax_1 + a'y_1 + a''z_1 \\ y = bx_1 + b'y_1 + b''z_1 \\ z = cx_1 + c'y_1 + c''z_1 \end{array}\right\}\ldots(3), \quad \left.\begin{array}{l} l = al_1 + a'm_1 + a''n_1 \\ m = bl_1 + b'm_1 + b''n_1 \\ n = cl_1 + c'm_1 + c''n_1 \end{array}\right\}\ldots(4).$$

Between the 9 quantities a, b, c; a', b', c'; a'', b'', c'', we have the 6 relations

[273] $$\left.\begin{array}{l} a'a''+b'b''+c'c''=0 \\ a''a+b''b+c''c=0 \\ aa'+bb'+cc'=0 \end{array}\right\}\ldots(5), \quad \left.\begin{array}{l} a^2+b^2+c^2=1 \\ a'^2+b'^2+c'^2=1 \\ a''^2+b''^2+c''^2=1 \end{array}\right\}\ldots(6).$$

By the substitution of the values of l, m, n from formulæ (4), (2) takes the form

$$Pl_1^2 + Qm_1^2 + Rn_1^2 + 2P'm_1n_1 + 2Q'l_1n_1 + 2R'l_1m_1 \ldots (7),$$

the quantities PQR, $P'Q'R'$ being functions of

$$abc, \quad a'b'c', \quad a''b''c'', \quad ABC, \quad fgh.$$

If we represent $\int x_1^2 dm$, &c. by A', &c., this expression is equivalent to

$$A'l_1^2 + B'm_1^2 + C'n_1^2 + 2f'm_1n_1 + 2g'l_1n_1 + 2h'l_1m_1 \ldots (8);$$

we thus find $f' = P'$, that is, $\int y'z'\, dm$ expressed in terms of

$$abc\ a' \ldots A \ldots.$$

and therefore, if the new axes are so chosen that $P'Q'R'$, the coefficients of the products, vanish in the transformed expression (2), these new axes will possess the property of making

$$\int x_1y_1\, dm = 0, \quad \int x_1z_1\, dm = 0, \quad \int y_1z_1\, dm = 0,$$

and are consequently principal axes of rotation.

(2) In finding the axes of elasticity of a crystal, we require a transformation of coordinates which will make

$$\frac{d^2U}{dx_1dy_1} = 0, \quad \frac{d^2U}{dx_1dz_1} = 0, \quad \frac{d^2U}{dy_1dz_1} = 0,$$

U being a known function of x, y, and z: this amounts to finding the linear substitution for dx, dy, dz, which will cause the terms involving dx_1dy_1, dx_1dz_1, dy_1dz_1 to vanish in the transformed expression for

$$\frac{d^2U}{dx^2}dx^2 + \frac{d^2U}{dy^2}dy^2 + \frac{d^2U}{dz^2}dz^2 + 2\frac{d^2U}{dy\,dz}dy\,dz + 2\frac{d^2U}{dx\,dz}dx\,dz$$
$$+ 2\frac{d^2U}{dx\,dy}dx\,dy.$$

(3) The well-known formulæ for the transformation of coordinates, contain either three arbitrary quantities independent of each other, or nine arbitrary quantities, with six relations between them: the use of the first set is embarrassing, from the complexity of the expressions and the want of any evident law in their formation; and even when the transformation is made by two successive steps, the calculations, though less embarrassing, are still, as may be seen from a preceding article in our present Number, both tedious and awkward.

(4) In making use of the symmetrical formulæ, the method which first presents itself is to substitute the expressions for the variables, and perform all the multiplications indicated. But the squares and products of the six trinomials give complex expressions. The conditions to be fulfilled [274] give equations involving the nine arbitrary quantities, and it is not easy to see how the eliminations are to be effected, and the value of each determined.

(5) This difficulty may be avoided in different ways. In the reduction of the equation of surfaces of the second order, by the consideration of diametral planes which are perpendicular to their chords, the transformation is performed by two successive operations. The consideration, that at the extremities of the principal axes, if such exist, the tangent plane must be perpendicular to the axes, and therefore the radius vector a maximum or minimum, leads to a very simple investigation, which will be found in the second Number of this Journal, p. 53.

The other problems which we have mentioned, are better solved from independent considerations, by which the conditions and the values of the coefficients are more easily obtained: it is, however, of importance to show, as we have done, that they depend essentially on the same transformation, and so account for the occurrence, in every method, of the final cubic equation. For these methods we may refer to the first Number of the Journal, pp. [4–35], and the latter has been inserted in the second edition of a *Treatise on Dynamics*, by Mr. Earnshaw, of St. John's College.

(6) Taking, then, the general form of a homogeneous function of the second degree in x, y, z,

$$Ax^2 + By^2 + Cz^2 + 2fyz + 2gxz + 2hxy \ldots\ldots (9).$$

This may be put under the form

$$(Ax + hy + gz)x + (hx + By + fz)y + (gx + fy + Cz)z \ldots (10),$$

and substituting for x, y, z the values (3), this becomes

$$\left.\begin{aligned}&(Lx_1 + L'y_1 + L''z_1)(ax_1 + a'y_1 + a''z_1)\\ +&(Mx_1 + M'y_1 + M''z_1)(bx_1 + b'y_1 + b''z_1)\\ +&(Nx_1 + N'y_1 + N''z_1)(cx_1 + c'y_1 + c''z_1)\end{aligned}\right\} \dots\dots (11).$$

In which
$$\left.\begin{aligned}L^{(m)} &= Aa^{(m)} + hb^{(m)} + gc^{(m)}\\ M^{(m)} &= ha^{(m)} + Bb^{(m)} + fc^{(m)}\\ N^{(m)} &= ga^{(m)} + fb^{(m)} + Cc^{(m)}\end{aligned}\right\} \dots\dots\dots\dots (12),$$

(11) may be thus put under the same form as (10),

$$(Px_1 + P'y_1 + P''z_1)\,x_1 + (P_1x_1 + P_1'y_1 + P_1''z_1)\,y_1 + (P_{''}x_1 + P_{''}'y_1 + P_{''}''z_1)\,z_1 \dots\dots (13).$$

In which $P_{(n)}{}^{(m)} = a^{(n)}L^{(m)} + b^{(n)}M^{(m)} + c^{(n)}N^{(m)} \dots (14)$;
and from the form of expression (12) it will be seen that

$$P_{(n)}{}^{(m)} = P_{(m)}{}^{(n)},$$

and therefore $P' = P_{,}$, $P'' = P_{''}$, $P_{,}'' = P_{''}'$;
and therefore the products will vanish if these quantities are each equal to zero.

[275] Now
$$\left.\begin{aligned}P &= aL + bM + cN\\ P_1 = 0 &= a'L + b'M + c'N\\ P_{''} = 0 &= a''L + b''M + c''N\end{aligned}\right\} \dots\dots (15).$$

If we multiply these by a, a', a'', and add,

$$aP = L, \text{ and therefore } = Aa + hb + gc,$$

$$\left.\begin{aligned}&\text{or} \quad (A - P)\,a + hb + gc = 0.\\ &\text{If we multiply by } b, b', b'', c, c', c'', \text{ we shall get}\\ &ha + (B - P)\,b + fc = 0,\\ &ga + fb + (C - P)\,c = 0,\end{aligned}\right\} \dots (16),$$

and eliminating a, b and c by cross-multiplication,

$$(A - P)(B - P)(C - P) - f^2(A - P) - g^2(B - P) - h^2(C - P) + 2fgh = 0$$
$$= (\text{say})\ U \dots\dots (17),$$

(7) This cubic equation, which constantly occurs in mechanics and geometry, has all its roots real. This will be evident if we put it under the form

$$\{(A - P)(B - P) - h^2\}\,\{(A - P)(C - P) - g^2\} - \{(P - A)f + gh\}^2 = 0,$$

by introducing the factor $(A - P)$.

Either of the two first factors, equated to zero, will give two real values of $A-P$, one positive the other negative, since the last terms are essentially negative; call these $-\rho$ and σ.

If in the left-hand side of the equation,

for $A - P$ we substitute	$-\infty$, the result is	$+$,
. . . .	$-\rho$, . .	$-$,
. . . .	0, . .	0,
. . . .	σ, . .	$-$,
. . . .	$+\infty$, . .	$+$.

Hence there is one real root $< -\rho$, another $> \sigma$; and as there is one $= 0$ between $-\rho$ and σ, and no change of sign, there must also be another; and hence all the roots are real.

(8) We should get the same cubic in finding the values of $a'b'c'$ and $a''b''c''$, as we evidently ought, since there is nothing to determine which of the three directions should be taken for the axis of x, or y, or z.

Let the three roots be P_1, P_2, P_3; it is evident we must take different roots according as we wish to find one of the sets abc, $a'b'c'$, or $a''b''c''$.

(9) To find the actual values of a, b, c, eliminate b between the first and second, and then between the second and third of equations (16): we thus get

$$a\{(A-P)(B-P)-h^2\} = c\{fh - g.(B-P)\}, \quad [276]$$
$$a\{fh - g\,.\,(B-P)\} = c\{(B-P)(C-P)-f^2\},$$

whence $$\frac{a^2}{(B-P)(C-P)-f^2} = \frac{c^2}{(A-P)(B-P)-h^2}$$
$$= \frac{b^2}{(A-P)(C-P)-g^2} = (\text{say})\ \mu.$$

Hence $$a^2+b^2+c^2 = \mu\,\{(A-P)(B-P)+(A-P)(C-P)$$
$$+(B-P)(C-P)-f^2-g^2-h^2\}$$
$$= -\mu\,\frac{dU}{dP};$$

and if for P we put P_1, one of the roots,

$$1 = \mu\,.\,(P_1-P_2)(P_1-P_3);$$

therefore $$a^2 = \frac{(B-P)(C-P)-f^2}{(P_1-P_2)(P_1-P_3)},$$

with similar expressions for b'^2, c'^2, a'^2, &c.

(10) The actual values of a, b, c are seldom required; in general it is sufficient to assure ourselves of the possibility of the transformation. The preceding investigation shows that it is always possible, and accounts for the constant occurrence of the equation $U = 0$, whenever the rectangularity of three lines is to be expressed.

A. S.

INVESTIGATION OF THE TENDENCY OF A BEAM TO BREAK WHEN LOADED WITH WEIGHTS.

In the following elementary Statical Problem, a discontinuous function presents itself, which admits of a very simple geometrical representation.

A beam, whose length is l, is supported horizontally at its extremities A and B. At distances a and b from these extremities is placed a weight W; and it is required to find the *tendency* of the beam to break at any point.

Let A and B represent the pressures at the extremities. Take $AP = x$, and suppose the weight of the beam to be neglected.

The tendency to break will be found by supposing one end of the beam to be fixed, say built into a wall, and finding the force tending to *turn* the other end about the point in consideration: we have thus, calling the tendency to break at P, T_x,

[277] $$T_x = Ax, \text{ or } = B(l-x) - W(a-x),$$

$$= \frac{Wb}{l} \cdot x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

It is evident that there is no tendency to break at the extremities; accordingly we find from the expression for T_x, $T_0 = 0$, but for B we have $T_l = Wl$, which it is evident cannot be the case. In fact, the function representing the tendency to break at any point is *discontinuous*, and changes its form at the point W.

If x be greater than a, we find

$$T_x = Ax - W(x-a) = B(l-x),$$

$$= \frac{Wa}{l}(l-x) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When $x = a$, the formulæ (1) and (2) agree in giving $T_a = \dfrac{Wab}{l}$.

If we take a line WC perpendicular to the beam to represent this, and join CA and CB, the tendency to break at any point P is represented by the ordinate to the broken line ACB.

We may represent the value of T_x, analytically, by help of discontinuous factors, whose values are unity or zero as x is greater or less than a.

$$\frac{1}{1+0^{a-x}} \text{ and } \frac{1}{1+0^{x-a}}$$

are of this nature, and have the values $1, \tfrac{1}{2}, 0$, as $x - a$ or $a - x$ are respectively positive, zero, or negative; so that

$$T_x = \frac{W}{l} \cdot \left\{\frac{bx}{1+0^{a-x}} + \frac{a(l-x)}{1+0^{x-a}}\right\},$$

$$= \frac{W}{l} \cdot \frac{1}{0^a + 0^x} \left\{bx\, 0^x + a(l-x)\, 0^a\right\}.$$

If we suppose the section and density of the beam y and ρ to vary according to any law; then the tendency to break is

$$T_x = Ax - g\int_0^x \rho y (x-z)\, dz;$$

when $x = l$, $T_l = 0$,

$$0 = Al - g\int_0^l \rho y (l-z)\, dz;$$

$$\therefore\ T_x = g\,\frac{x}{l}\int_0^l \rho y (l-z)\, dz - g\int_0^x \rho y (x-z)\, dz,$$

$$= g\,\frac{l-x}{l}\int_0^x \rho yz\, dz + g\,\frac{x}{l}\int_x^l \rho y (l-z)\, dz.$$

If the beam is loaded with n weights $W_1, W_2 \dots W_n$, placed at distances $x_1 \dots x_n$ from A, and x lies between x_r and x_{r+1}, then it will be seen that we must add to the value of T_x,

$$\frac{l-x}{l}\sum_1^r W_s x_s + \frac{x}{l}\sum_{r+1}^r W_s (l - x_s). \qquad [278]$$

The first part of the expression will represent a curve, which may be constructed on one side of AB; the second will represent the sides of a polygon, constructed also upon AB; and the tendency to break will be given by the sum of the ordinates to the two lines.

We may, as before, represent the complete value of T_x, which will be

$$T_x = g\frac{l-x}{l}\int_0^l \frac{\rho yz\,dz}{1+0^{x-z}} + g\frac{x}{l}\int_0^l \frac{\rho y(l-z)\,dz}{1+0^{z-x}}$$

$$+\frac{l-x}{l}\sum_1^n \frac{W_s x_s}{1+0^{x-x_s}} + \frac{x}{l}\sum_1^n \frac{W_s(l-x_s)}{1+0^{x_s-x}}$$

$$=\frac{g}{l}\int_0^l \frac{\rho y}{0^x+0^z}\cdot\{(l-x)\,z0^z + x(l-z)\,0^x\}\,dz$$

$$+\frac{1}{l}\sum_l^n \frac{W_s}{0^x+0^{x_s}}\{(l-x)\,x_s 0^{x_s} + x(l-x_s)\,0^x\}.$$

H. T.

MATHEMATICAL NOTES.

1.* The Solution of the Linear Equation of the n^{th} order contained in the First Number of this Journal, fails in a case which is not there adverted to. The instance alluded to is, when the equation $f\left(\frac{d}{dx}\right) = 0$ has equal roots. In this case we need not reject the general method, but in applying it we must adopt the following modification.

Suppose the equation to have r roots equal to a, and all the rest unequal; then $f\left(\frac{d}{dx}\right)y = X$ is equivalent to

$$\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\dots\left(\frac{d}{dx}-a_{n-r}\right)\left(\frac{d}{dx}-a\right)^r y = X;$$

whence we may obtain, by the general method,

$$\left(\frac{d}{dx}-a\right)^r y = \frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}\,Xdx\right)}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_{n-r})}$$

$$+\frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}\,Xdx\right)}{(a_3-a_1)(a_2-a_3)\dots(a_2-a_{n-r})}$$

$$+\dots+\frac{\varepsilon^{a_{n-r}x}\left(\int\varepsilon^{-a_{n-r}x}\,Xdx\right)}{(a_{n-r}-a_1)(a_{n-r}-a_2)\dots(a_{n-s}-a_{n-r-1})}.$$

* From a Correspondent.

Putting the second side equal to X_1, we have

$$y = \left(\frac{d}{dx} - a\right)^{-r} X_1$$
$$= \varepsilon^{ax} \int^r dx^r \, \varepsilon^{-ax} X_1.$$

The complementary function to be added will be found by making $X = 0$, when we get

$$y = \varepsilon^{ax} \int^r dx^r \, 0$$
$$= \varepsilon^{ax} (C_1 x^{r-1} + C_2 x^{r-2} + \&c. + C_{r-1}). \qquad \lambda.$$

2. *Addendum to Art.* 4. *No. V.*—The following transformation may be added to those given in page [217]. Take the series

$$S = AA_1 + BB_1 \, x + CC_1 x^2 + \&c.,$$

and let

$$B = DA, \quad C = D^2 A, \ \&c.$$
$$B_1 = D_1 A_1, \quad C_1 = D_1^{\,2} A_1, \ \&c.;$$

then the expression becomes

$$S = (1 + xDD_1 + x^2 D^2 D_1^{\,2} + \&c.)\, AA_1$$
$$= (1 - xDD_1)^{-1} AA_1 = \{1 - xD\,(1 + \Delta_1)\}^{-1} AA_1$$
$$= \{1 - D\,(x + x\Delta_1)\}^{-1} AA_1 = \varepsilon^{x\Delta_1 \frac{d}{dx}} (1 - xD)^{-1} AA_1.$$

Now $(1 - xD)^{-1} A = A + Bx + Cx^2 + \&c. = X$ suppose; therefore the series becomes

$$S = \varepsilon^{x\Delta_1 \frac{d}{dx}} A_1 X = A_1 X + \frac{x\Delta A_1}{1} \frac{dX}{dx} + \frac{x^2 \Delta^2 A_1}{1.2} \frac{d^2 X}{dx^2} + \&c.$$

ϕ.

3. The investigation of the locus of a straight line, which rests constantly on three given straight lines, as given by Leroy in page 98 of his *Analytical Geometry*, may be much simplified by using the symmetrical equations to the straight line. Taking the same notation as Leroy, the equations to the directrices are

$$\left.\begin{matrix} x = +\,\alpha \\ y = -\,\beta \end{matrix}\right\} \dots (1), \qquad \left.\begin{matrix} z = +\,\gamma \\ x = -\,\alpha \end{matrix}\right\} \dots (2), \qquad \left.\begin{matrix} y = +\,\beta \\ z = -\,\gamma \end{matrix}\right\} \dots (3),$$

and the equations to the generatrix are

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n};$$

between which we have to eliminate a, b, c, l, m, n.

[280] The conditions that the generatrix shall pass through (1), (2), (3) respectively, give us the equations

$$\frac{x-a}{l}=\frac{y+\beta}{m},\qquad \frac{z-\gamma}{n}=\frac{x+a}{l},\qquad \frac{y-\beta}{m}=\frac{z+\gamma}{n}.$$

Multiplying these equations together, l, m, n are eliminated, and we have

$$(x-a)(y-\beta)(z-\gamma)=(x+a)(y+\beta)(z+\gamma),$$

which reduces to

$$ayz+\beta xz+\gamma xy+a\beta\gamma=0,$$

the equation to the hyperboloid of one sheet.

ϕ.

4.* I propose to exhibit the solution of the partial differential equation

$$\frac{d^nz}{dx^n}+A_1\frac{d^nz}{dx^{n-1}dy}+A_2\frac{d^nz}{dx^{n-2}dy}+\ldots+A_n\frac{d^nz}{dy^n}=V,$$

where $A_1, A_2 \ldots A_n$ are constant, and V any function of x and y. The method I shall follow will bear a strong analogy to that applied to the linear equation between two variables; but it will be requisite to go into it more fully than might at first sight appear necessary.

Separating the symbols, we shall have

$$z=\left(\frac{d}{dx}-a_1\frac{d}{dy}\right)^{-1}\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\frac{d}{dy}\right)^{-1}V.$$

Where $a_1, a_2 \ldots a_n$ are the roots of the equation,

$$a^n+A_1a^{n-1}+\ldots+A_n=0\ldots(a),$$

therefore

$$z=\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1}\left(\frac{d}{dx}-a_3\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\frac{d}{dy}\right)^{-1}\varepsilon^{a_1x\frac{d}{dy}}\int\varepsilon^{-a_1x}Vdx$$

$$=\left(\frac{d}{dx}-a_3\frac{d}{dy}\right)^{-1}\ldots\left(\frac{d}{dx}-a_n\right)^{-1}\left\{\frac{\varepsilon^{a_1x\frac{d}{dy}}\int\varepsilon^{-a_1x\frac{d}{dy}}Vdx}{(a_1-a_2)\frac{d}{dy}}+\frac{\varepsilon^{a_2x\frac{d}{dy}}\int\varepsilon^{-a_2x\frac{d}{dy}}Vdx}{(a_2-a_1)\frac{d}{dy}}\right\}\ldots(1);$$

* From a Correspondent.

therefore [281]

$$\frac{dz}{dy} = \left(\frac{d}{dx} - a_3 \frac{d}{dy}\right)^{-1} \cdot\cdot \left(\frac{d}{dx} - a_n \frac{d}{dy}\right)^{-1}$$

$$\left(\frac{\varepsilon^{a_1 x \frac{d}{dy}} \int \varepsilon^{-a_1 x \frac{d}{dy}} V dx}{a_1 - a_2} + \frac{\varepsilon^{a_2 x \frac{d}{dy}} \int \varepsilon^{-a_2 x \frac{d}{dy}} V dx}{a_2 - a_1}\right) \ldots . (2);$$

and proceeding in this manner we shall finally arrive at the equation

$$\frac{d^{n-1} z}{dy^{n-1}} = \frac{\varepsilon^{a_1 x \frac{d}{dy}} \int \varepsilon^{-a_1 x \frac{d}{dy}} V dx}{(a_1 - a_2)(a_1 - a_3) \cdot\cdot (a_1 - a_n)}$$

$$+ \frac{\varepsilon^{a_2 x \frac{d}{dy}} \int \varepsilon^{-a_2 x \frac{d}{dy}} V dx}{(a_2 - a_1)(a_2 - a_3) \ldots (a_2 - a_n)} + \cdot\cdot + \frac{\varepsilon^{a_n x \frac{d}{dy}} \int \varepsilon^{-a_n x \frac{d}{dy}} V dx}{(a_n - a_1)(a_n - a_2) \ldots (a_n - a_{n-1})} \cdot\cdot (n).$$

Let $$\int \varepsilon^{-a_1 x \frac{d}{dy}} V dx = \varepsilon^{-a_1 x} V_1 + f_1(y),$$

where f denotes an arbitrary function; and let the denominators of the above fractions be represented by $\frac{1}{C_1}$, $\frac{1}{C_2} \ldots \frac{1}{C_n}$; then

$$\frac{d^{n-1} y}{dy^{n-1}} = C_1 V_1 + C_2 V_2 + \ldots + C_n V_n + C_1 \varepsilon^{a_1 x \frac{d}{dy}} f_1(y)$$

$$+ C_2 \varepsilon^{a_2 x \frac{d}{dy}} f_2(y) + \ldots + C_n \varepsilon^{a_n x \frac{d}{dy}} f_n(y);$$

and observing that $\varepsilon^{a_1 x \frac{d}{dy}} f(y)$ is equivalent to $f(y + a_1 x)$, we shall have, integrating,

$$z = C_1 \int V_1 dy^{n-1} + C_2 \int V_2 dy^{n-1} + \ldots + C_n \int V_n dy^{n-1}$$

$$= \phi_1 (y + a_1 x) + \phi_2 (y + a_2 x) + \ldots + \phi_n (y + a_n x),$$

where ϕ denotes an arbitrary function.

It is most important to observe, that in integrating the expression for $\frac{d^{n-1} z}{dy^{n-1}}$ we are not at liberty to introduce any arbitrary functions of x, as will appear from the following considerations. In passing from equation (1) to equation (2), no arbitrary function of x is lost; and a similar remark applies to each succeeding pair of the equations (1), (2)...(n): so that, in passing from the expression for z to that for $\frac{d^{n-1} z}{dy^{n-1}}$ no arbitrary function of x is lost, and consequently, in passing

back again from the expression for $\frac{d^{n-1}z}{dy^{n-1}}$ no arbitrary function of x must be introduced.

If the roots of the equation (a) are not all unequal, there must be a modification of the above process, similar to that suggested above in the case of the linear equation between two variables.

[282] 5. To find the product of the differential factors

$$\frac{d}{dx}\left(\frac{d}{dx}-1\right)\left(\frac{d}{dx}-2\right)\dots\left\{\frac{d}{dx}-(n-1)\right\}.$$

By the theorem in page 25, we have generally

$$\left(\frac{d}{dx}-a\right)^n = \varepsilon^{ax}\left(\frac{d}{dx}\right)^n \varepsilon^{-ax};$$

therefore $$\left(\frac{d}{dx}-1\right)\frac{d}{dx} = \varepsilon^{x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

$$\left(\frac{d}{dx}-2\right)\left(\frac{d}{dx}-1\right)\frac{d}{dx} = \varepsilon^{2x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

$$\left(\frac{d}{dx}-3\right)\left(\frac{d}{dx}-2\right)\left(\frac{dx}{dx}-1\right)\frac{d}{dx} = \varepsilon^{3x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx}\varepsilon^{-x}\frac{d}{dx},$$

and so on, so that

$$\left\{\frac{d}{dx}-(n-1)\right\}\left\{\frac{d}{dx}-(n-2)\right\}\dots\left(\frac{d}{dx}-1\right)\frac{d}{dx}$$

$$= \varepsilon^{(n-1)x}\frac{d}{dx}\varepsilon^{-x}dx \dots (n \text{ factors}),$$

$$= \varepsilon^{nx}\left(\varepsilon^{-x}\frac{d}{dx}\right)^n,$$

$$= y^n\left(\frac{d}{dy}\right)^n \text{ if } x = \log y.$$

This theorem may be otherwise expressed—that

$$y^n\left(\frac{d}{dy}\right)^n = y\frac{d}{dy}\left(y\frac{d}{dy}-1\right)\left(y\frac{d}{dy}-2\right)\dots\left\{y\frac{d}{dy}-(n-1)\right\}.$$

This affords an easy demonstration of a problem in the Senate-House papers for 1839.

Since $x^n = (x+0)^n = (1+\Delta)^x\, 0^n$

$$= x\Delta 0^n + \frac{x(x-1)}{1.2}\Delta^2 0^n + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 0^n + \dots$$

for x put $y\,\frac{d}{dy}$; then

$$\left(y\,\frac{d}{dy}\right)^n = y\,\frac{d}{dy}\,\Delta 0^n + y^2\left(\frac{d}{dy}\right)^2\frac{\Delta^2 0^n}{1.2} + y^3\left(\frac{d}{dy}\right)^3\frac{\Delta^3 0^n}{1.2.3} + \dots$$

which is a general formula for changing the independent variable from x to ε^x.

ϕ.

END OF VOL. I.

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